

Report No. RD- 65-38

FINAL REPORT

AIR TRAFFIC SURVEILLANCE SATELLITES A MATHEMATICAL MODEL FOR ACCURACY AND COVERAGE



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FEDERAL AVIATION AGENCY
Systems Research & Development Service
Atlantic City, New Jersey

FINAL REPORT

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A MATHEMATICAL MODEL FOR ACCURACY AND COVERAGE

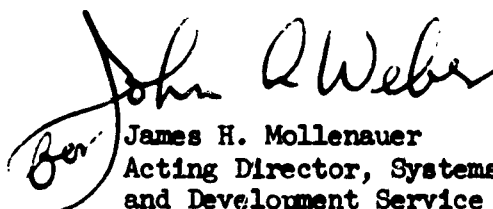
REPORT NO. RD-65-38

Prepared by:

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APRIL 1965

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A handwritten signature in black ink, appearing to read "John Q. Weber". The signature is written in a cursive, flowing style. To the left of the main signature, there is a smaller, less legible handwritten mark that appears to be "Ber".

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MODEL FOR ACCURACY AND COVERAGE by John J. Kulik, Final
Report, April 1965, 59 pp., incl. 32 illus., 2 tables.
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ABSTRACT

The theoretical accuracy and coverage of the two most promising air traffic surveillance satellite techniques--a two-satellite multi-rho technique and a single satellite rho-theta-theta technique--are determined by the application of a mathematical model used to derive error isograms for several combinations of satellite positions and range and angle measurement error assumptions. The complete derivation and discussion of the model employed is included, since the model is sufficiently general to apply to a great number of situations involving position determination other than air traffic surveillance. Some conclusions as to the relative merits of the two techniques studied are given, and recommendations are made for future activity.

INTRODUCTION

The Systems Research and Development Service has the responsibility of studying innovations in operational techniques and introducing new technical advances for future implementation in the air traffic control system of the Federal Aviation Agency and is making studies of Navigation, Communication, and Surveillance systems for use on the North Atlantic Air Route for the post-1970 era. A report by the Communication Group of the University of Michigan Institute of Science and Technology [1]* recommended a synchronous satellite communication system for Air Traffic Control over the North Atlantic Air Route. Of all the systems considered, the satellite system was deemed most reliable and economically feasible for implementation in the post-1970 time period. On the basis of this recommendation, Systems Research and Development Service has been making studies (in-the-house and under contract) pertinent to the design and synthesis of a synchronous satellite communication subsystem for over-ocean communications. Preliminary studies of propagation and coverage factors show that the VHF (Very High Frequency) Aeronautical Communication Band can be used, and that a two-satellite synchronous orbit configuration is preferable to three-, four- and five-satellite configurations [2].

In a concurrent study of position surveillance systems for Air Traffic Control over the North Atlantic Ocean by the Navigation and Control Laboratory, Institute of Science and Technology at the University of Michigan, the recommendation was made that a position data acquisition system in the post-1970 time period consist of a two-satellite synchronous orbit configuration using a multiple ranging technique. Two systems embodying this technique have been studied. The first, identified by the acronym LOCATES (Location of Air Traffic Enroute by Satellite) was proposed by the University of Michigan as part of their study [3]. The second was a two-satellite synchronous altitude system for Coverage of the North Atlantic area, proposed by the General Electric Company as an alternate to the multi-satellite, medium altitude, worldwide system of their final report

* All references will be listed at the end of this report.

on a study of satellites for navigation for the National Aeronautics and Space Administration [4].

The Westinghouse Electric Corporation's Defense and Space Center under Contract NASw-785 sponsored by the National Aeronautics and Space Administration has proposed a navigation satellite system (which can be used for a position data acquisition function) consisting of a single synchronous satellite using a combination of ranging and angle measurements stationed above the equator between 30° and 40° West Longitude [5].

Although each of the study efforts just mentioned presented some data on the positional errors to be expected in operational systems, the assumptions made, the mathematical methods employed and the means of presenting results differed for each study. As a consequence, it has been difficult to make valid comparisons among the systems. Therefore, an in-house task was established to construct a general mathematical model which would relate the distribution of error in determining the position of aircraft by various position finding systems to the distributions of assumed input errors (bias errors are assumed removable) involved in measuring angles and ranges between station and aircraft thereby determining the theoretical system quality of any navigation and surveillance systems which might be hypothesized.

The body of this report consists of two parts. The first part (General Theory) includes a brief, non-mathematical discussion of generic concepts, which unify position determination systems, in terms of the geometry of quadric surfaces. A discussion of the problem of system ambiguities and the reasons for the existence of multiple values is given. A general Probabilistic Position Determination System Model is then constructed and developed which can be applied to specific navigation and surveillance systems. The model is a first approximation to the real physical systems under consideration which allows using the tools of probability and statistics in their most fundamental aspects, rather than the more esoteric nuances of these mathematical formalisms. For example, it is assumed that the random errors are normally distributed without any attempt to justify using normal statistics by introducing the Central Limit Theorem since it is assumed beyond the scope of the report to do so.

In the development of the model, coordinate transformations are made by introducing matrix methods which simplify the notation and allow a clearer picture of the methodology involved. The matrix methods also give a better picture of the transformation of axes which is necessary in a model to be used for computational purposes.

In the second part of the body of this report (Applications), the mathematical model from the General Theory is applied first to the two-satellite system (LOCATES) and then to the one-satellite system. These two systems are representative of air traffic surveillance satellite techniques being studied today. This section includes error isograms for the two systems, compares them with respect to positional accuracy and coverage, and makes some observations of the relative state-of-the-art of the two systems.

GENERAL THEORY

1. Fundamental Concepts. The quantities measured by position determination systems (directly or indirectly) are distances and angles. The measurements are made relative to known or measured positions and orientations. The generic terms RHO and THETA are applied to distance (range) and angle measurements, respectively, and are used for system description. For example, a RHO-RHO-THETA system measures two distances and one angle. From each measurement made by a system, a quadric surface (a surface in three-dimensional space) is generated. A position in three-dimensional space is determined by the intersection of three quadric surfaces. The quadric surfaces generated by measurements of position determination systems are limited to spheres generated by RHO measurements; cones generated by THETA measurements;* and planes generated by THETA measurements. (Strictly speaking, the plane is not a quadric surface since the equation of a plane is linear but the linear equation is a degenerate form of the general equation of a quadric surface.) When a position is to be determined, measurements are made and three equations are given:

$$\left. \begin{array}{l} M_1 = C_1 \\ M_2 = C_2 \\ M_3 = C_3 \end{array} \right\} \quad (1)$$

where the M_i are measures of either ranges or angles and the C_i are constants. The equations (1) are the equations of three quadric surfaces in their most general form. They can be transformed to the more familiar Cartesian coordinates as:

*The cones approximate hyperboloids for some systems.

$$\left. \begin{aligned} F(x, y, z) &= 0 \\ G(x, y, z) &= 0 \\ H(x, y, z) &= 0 \end{aligned} \right\} \quad (2)$$

where

$$\begin{aligned} F(x, y, z) &= a_{11} x^2 + 2 a_{12} x y + 2 a_{13} x z \\ &+ a_{22} y^2 + 2 a_{23} y z + a_{33} z^2 \\ &+ 2 a_{14} x + 2 a_{24} y + 2 a_{34} z + a_{44} \end{aligned}$$

with the a_{ijk} = constants.

$G(x, y, z)$ and $H(x, y, z)$ have the same form but the constants a_{ijk} are different. In order to determine a position in x, y , and z space, it is necessary to solve the equations (2) simultaneously for x_1, y_1, z_1 , the coordinates of the position. Because the equations are quadratic (the independent variables are second degree) solving them simultaneously yields, in general, eight positions. For specific surfaces and a particular choice of origin (usually the center of system symmetry) they are:

$$\begin{aligned} P_1 &= (x_1, y_1, z_1) \\ P_2 &= (x_1, y_1, -z_1) \\ P_3 &= (x_1, -y_1, z_1) \\ P_4 &= (x_1, -y_1, -z_1) \\ P_5 &= (-x_1, y_1, z_1) \\ P_6 &= (-x_1, y_1, -z_1) \\ P_7 &= (-x_1, -y_1, z_1) \\ P_8 &= (-x_1, -y_1, -z_1) \end{aligned}$$

The number of points found for any specific position determination system depends upon the surfaces generated by the system. A RHO-RHO-RHO system, for example, generates three spheres. The intersection of two of the spheres is a circle and the intersection of the circle with the third sphere yields two points. Systems can be engineered to determine a unique point in space. For example, a tracking radar (RHO-THETA-THETA) measures a range and two angles:

$$\left. \begin{aligned} R &= R_1 \\ \theta &= \theta_1 \\ \phi &= \phi_1 \end{aligned} \right\} \quad (3)$$

where R is range, θ is elevation angle, ϕ is azimuth angle, and R_1, θ_1, ϕ_1 , are constants. The equations (3) are the equations of a sphere, cone, and plane, respectively. The point (R_1, θ_1, ϕ_1) is the intersection of the quadric surfaces as shown in Fig. 1.

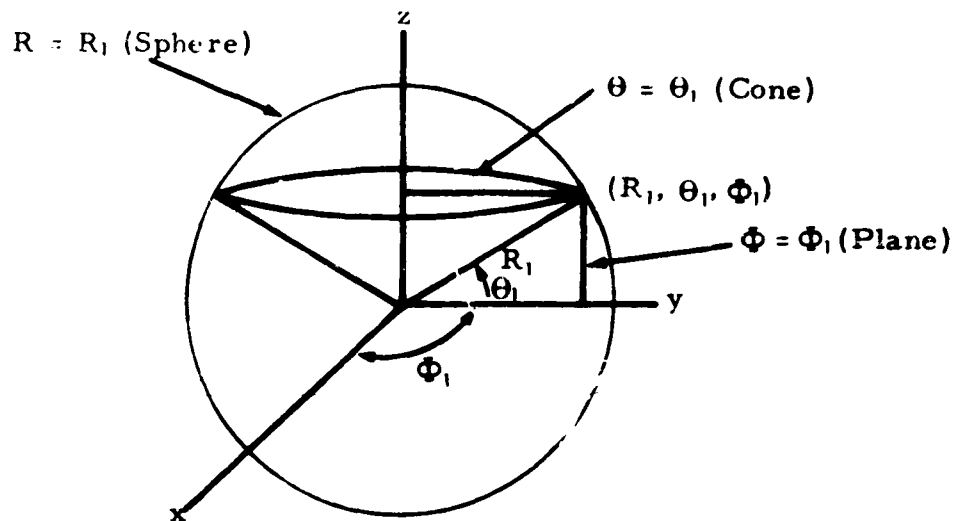


FIG. 1 QUADRIC SURFACES ASSOCIATED WITH A TRACKING RADAR SHOWING THEIR INTERSECTION AT (R_1, θ_1, ϕ_1) .

The point (R_1, θ_1, ϕ_1) is unique because of system conditions. $R = -R_1$ yields the same sphere as $R = R_1$ but the system always measures $R \geq 0$. Since the system scans the half space above earth, it only measures $\theta \geq 0$. The system is set up so that:

$$\Phi = \Phi_1 \pm \Phi_1 + 180^\circ$$

i.e., it has a unidirectional beam of radiation. Because of these auxiliary conditions the tracking radar determines a unique point. If any coordinate transformations are used on the tracking radar system equations, the auxiliary conditions must be carried along to avoid ambiguities. The problem of multiple values of a system can usually be handled quite easily, but some consideration of them must be made in any system design.

2. Probabilistic System Model. It has been shown that a position in space is determined by finding the intersection of three quadric surfaces. When a position is to be determined, measurements are made giving the values M_1, M_2, M_3 . These measurements are used to generate a set of equations which can be solved simultaneously for the position coordinates. M_1, M_2 , and M_3 are rarely exact measures (the probability that they are exact is zero) since random errors can never be completely excluded from them. Since we are interested in finding

the probability that a measured position is within some linear distance of the true position, we assume that the i 'th measurement of a position includes linear errors $\epsilon_{M_1}^i$, $\epsilon_{M_2}^i$ and $\epsilon_{M_3}^i$ in the measures of M_1 , M_2 , and M_3 , respectively. If the measurement is a range measurement, the error is along the vector from the station to the position, and if the measurement is an angle, the error is linear and normal to the vector from the station to the position. The angular error in an angle measurement can always be determined by dividing the linear error by the magnitude of the vector from the station to the position. If a new measurement (the j 'th) is made on the system with no change in the actual position, a new set of linear errors results $\epsilon_{M_1}^j$, $\epsilon_{M_2}^j$, and $\epsilon_{M_3}^j$ where, in general,

$$\begin{aligned}\epsilon_{M_1}^i &\neq \epsilon_{M_1}^j \\ \epsilon_{M_2}^i &\neq \epsilon_{M_2}^j \\ \epsilon_{M_3}^i &\neq \epsilon_{M_3}^j\end{aligned}$$

If a whole series of measurements is made with the actual position unchanged, a distribution of position determinations about the true position is given. From the distribution a probability density function is derivable. When the probability density function for a region of space is known, it is possible to determine the probability that a position determination will fall into any portion of the region. If, for example, the probability density function throughout some given volume of space V is p , the probability P that a position determination is in V is given by:

$$P = \iiint_V p \, dV \quad (4)$$

The model to be constructed in this report can be applied to specific position determination systems and the result of the application will give the probability that a position determination is in some given region of space about the true position.

The integration of equation (4) is easiest to perform when the probability density function p and the volume V over which the integration is to be made are geometrically compatible. For example, if p is a spherically symmetrical function, and the integration is over a spherical volume whose center is the center of symmetry, the problem has geometrical compatibility and is in its most suitable mathematical form. For a system where errors are considered random and normally distributed, the natural geometrical configuration is the ellipsoid. That is, the surfaces on which the probability density function is constant are homothetic ellipsoids about the true position as the center of gravity of the

distribution when the errors have zero means. Thus, the ellipsoids are called trivariate normal equi-probability error density ellipsoids. Although ellipsoids can be described in any arbitrary orthogonal coordinate set, the equations are generally complicated and difficult to work with so it is usually best to work in the coordinate axes which are coincident with the principal axes of the ellipsoid. Since many real position determination systems are originally given in the natural physical set of coordinate axes which are generally non-orthogonal and not coincident with the principal axes of the system ellipsoids, it is desirable to transform from the original non-orthogonal set (M_1, M_2, M_3) to the orthogonal principal axes set (p_1, p_2, p_3). The systems whose original physical sets of coordinate axes do coincide with the principal axes sets are special simple cases included in the general mathematical model to be developed here. Since the original set of axes is assumed non-orthogonal, the transformation to principal axes is made in two steps. First, a transformation matrix, T , is developed which transforms errors from the given non-orthogonal set (M_1, M_2, M_3) to an arbitrary, but known, orthogonal set of axes (x_1, x_2, x_3). Second, a rotation matrix, R , is developed which rotates errors from the orthogonal set (x_1, x_2, x_3) to errors in the principal axes set (p_1, p_2, p_3). In mathematical form:

$$\begin{aligned}\vec{\epsilon}'_x &= T \vec{\epsilon}'_M \\ \vec{\epsilon}'_p &= R \vec{\epsilon}'_x\end{aligned}$$

The complete transformation is given by:

$$\vec{\epsilon}'_p = R T \vec{\epsilon}'_M$$

Before the T and R matrices are developed, the significance of having the errors in the principal axes coordinate set will be discussed. As has been mentioned previously, ellipsoids can be described in any arbitrary orthogonal coordinate set (x_1, x_2, x_3). However, the probability density function in the (x_1, x_2, x_3) set is given by:

$$p(\epsilon_{x_1}, \epsilon_{x_2}, \epsilon_{x_3}) = \frac{e^{-\frac{1}{2} Q(\epsilon_{x_1}, \epsilon_{x_2}, \epsilon_{x_3})}}{(2\pi)^{3/2} \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sqrt{|e|}}$$

where:

$$Q(\epsilon_{x_1}, \epsilon_{x_2}, \epsilon_{x_3}) = \frac{1}{|e|} \sum_{i,k} e_{ik} \frac{\epsilon_{x_i} \epsilon_{x_k}}{\sigma_{x_i} \sigma_{x_k}} \quad (i, k = 1, 2, 3)$$

$$|e| = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix}$$

$$\rho_{jk} = \frac{1}{\sigma_{x_j} \sigma_{x_k} n} \sum_{i=1}^N (\epsilon_{x_j}^i \epsilon_{x_k}^i) \quad \left\{ \begin{array}{l} j, k = 1, 2, 3 \\ j \neq k \end{array} \right\}$$

$$\rho_{jj} = 1 \quad (j = 1, 2, 3)$$

$$\vec{\epsilon}_x = T \vec{\epsilon}_M$$

It is evident that $p(\epsilon_{x_1}, \epsilon_{x_2}, \epsilon_{x_3})$ is a complicated expression and that the integration of equation (4),

$$P = \iiint_V p(\epsilon_{x_1}, \epsilon_{x_2}, \epsilon_{x_3}) dV,$$

would be quite difficult for any volume V . However, if the error components are rotated to the principal axes set (p_1, p_2, p_3) according to:

$$\vec{\epsilon}_p = R \vec{\epsilon}_x$$

then the correlation ρ_{jk} is:

$$\rho_{jk} = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases} \quad (j, k = 1, 2, 3) \quad (5)$$

Then:

$$|\rho| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$Q(\epsilon_{p_1}, \epsilon_{p_2}, \epsilon_{p_3}) = \frac{\epsilon_{p_1}^2}{\sigma_{p_1}^2} + \frac{\epsilon_{p_2}^2}{\sigma_{p_2}^2} + \frac{\epsilon_{p_3}^2}{\sigma_{p_3}^2}$$

$$\sigma_{p_j}^2 = \frac{1}{n} \sum_{i=1}^N (\epsilon_{p_j}^i)^2 \quad (j = 1, 2, 3)$$

The probability density function in the principal axes set becomes:

$$p(\epsilon_{p_1}, \epsilon_{p_2}, \epsilon_{p_3}) = \frac{e^{-\frac{1}{2} \left(\frac{\epsilon_{p_1}^2}{\sigma_{p_1}^2} + \frac{\epsilon_{p_2}^2}{\sigma_{p_2}^2} + \frac{\epsilon_{p_3}^2}{\sigma_{p_3}^2} \right)}}{(2\pi)^{\frac{3}{2}} \sigma_{p_1} \sigma_{p_2} \sigma_{p_3}}$$

a relatively simple expression. To find the probability that a position determination is in an ellipsoid whose axes are $c\sigma_{p_1}$, $c\sigma_{p_2}$, $c\sigma_{p_3}$ where c is a constant, evaluate the integral:

$$P_c = \frac{1}{(2\pi)^{3/2}} \iiint \frac{e^{-\frac{1}{2}\left(\frac{\epsilon_{p_1}^2}{\sigma_{p_1}^2} + \frac{\epsilon_{p_2}^2}{\sigma_{p_2}^2} + \frac{\epsilon_{p_3}^2}{\sigma_{p_3}^2}\right)} d\epsilon_{p_1} d\epsilon_{p_2} d\epsilon_{p_3}}{\sigma_{p_1} \sigma_{p_2} \sigma_{p_3}}$$

$$\begin{aligned} \text{Let } z_1 &= \frac{\epsilon_{p_1}}{\sigma_{p_1}} & dz_1 &= \frac{d\epsilon_{p_1}}{\sigma_{p_1}} \\ z_2 &= \frac{\epsilon_{p_2}}{\sigma_{p_2}} & dz_2 &= \frac{d\epsilon_{p_2}}{\sigma_{p_2}} \\ z_3 &= \frac{\epsilon_{p_3}}{\sigma_{p_3}} & dz_3 &= \frac{d\epsilon_{p_3}}{\sigma_{p_3}} \end{aligned}$$

Then

$$P_c = \frac{1}{(2\pi)^{3/2}} \iiint e^{-\frac{1}{2}(z_1^2 + z_2^2 + z_3^2)} dz_1 dz_2 dz_3$$

Transforming to spherical coordinates

$$\begin{aligned} z_1 &= r \sin \theta \cos \phi \\ z_2 &= r \sin \theta \sin \phi \\ z_3 &= r \cos \theta \end{aligned}$$

Then

$$z_1^2 + z_2^2 + z_3^2 = r^2$$

$$dz_1 dz_2 dz_3 = r^2 \sin \theta dr d\theta d\phi$$

$$P_c = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^c e^{-\frac{r^2}{2}} r^2 dr$$

where

$$\begin{aligned} c &= \left(\frac{\epsilon_{p_1}^2}{\sigma_{p_1}^2} + \frac{\epsilon_{p_2}^2}{\sigma_{p_2}^2} + \frac{\epsilon_{p_3}^2}{\sigma_{p_3}^2} \right)^{1/2} \\ P_c &= \frac{2}{\sqrt{2\pi}} \int_0^c e^{-\frac{r^2}{2}} r^2 dr \end{aligned}$$

Integrate by parts letting

$$\begin{aligned} u &= r & \text{and } dv &= r e^{-\frac{r^2}{2}} dr \\ \text{then} \\ du &= dr & \text{and } v &= -e^{-\frac{r^2}{2}} \end{aligned}$$

$$P_c = \frac{2}{\sqrt{2\pi}} \int_0^c u \, dv = \frac{2}{\sqrt{2\pi}} \left(u v \Big|_0^c - \int_0^c v \, du \right)$$

$$P_c = \frac{2}{\sqrt{2\pi}} \left(-c e^{-\frac{c^2}{2}} + \int_0^c e^{-\frac{r^2}{2}} dr \right)$$

$$P_c = 2 \left(\frac{1}{\sqrt{2\pi}} \int_0^c e^{-\frac{r^2}{2}} dr - \frac{c}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} \right)$$

$$P_c = 2 (A_c - cB_c)$$

where

$$A_c = \frac{1}{\sqrt{2\pi}} \int_0^c e^{-\frac{r^2}{2}} dr$$

$$A_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^c e^{-\frac{r^2}{2}} dr - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{r^2}{2}} dr$$

$$A_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^c e^{-\frac{r^2}{2}} dr - 0.5$$

$$B_c = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}$$

A_c and B_c are tabulated in many probability and statistics books [6]. A very short table of P_c values is given in Table I.

TABLE I

A SHORT TABLE OF VALUES OF PROBABILITY OF A POSITION
DETERMINATION BEING IN A ONE, TWO, OR THREE SIGMA
ELLIPSOID WHOSE CENTER OF GRAVITY IS THE TRUE POSITION

c	A _c	c B _c	P _c
1	.34134	.24197	.19874
2	.47725	.10798	.73854
3	.99865	.01329	.97072

Table I shows the probability that a position determination in a one, two, or three sigma ellipsoid is .19874, .73854, and .97072, respectively. The actual physical dimensions of the principal axes of the ellipsoid are given by $c\sigma_{p_1}$, $c\sigma_{p_2}$, $c\sigma_{p_3}$ (with $c = 1, 2, 3$, respectively) where the values of σ_{p_j} are to be determined from:

$$\sigma_{p_j}^2 = \frac{1}{n} \sum_{i=1}^N (\epsilon_{p_j}^i)^2 \quad (j = 1, 2, 3)$$

and $\epsilon_{p_j}^i$ ($j = 1, 2, 3$) are determined from:

$$\epsilon_p^i = R^T \epsilon_m^i$$

Since the orientations of the (M_1 , M_2 , M_3) axes are given and the T and R matrices will be developed, the ellipsoid orientation will also be determined.

In order to find the T matrix in:

$$\epsilon_x^i = T \epsilon_m^i$$

it is necessary to find the vector ϵ_m^i from the given components $\epsilon_{m_1}^i$, $\epsilon_{m_2}^i$, $\epsilon_{m_3}^i$. Assume that the errors are small and the distance between stations at which measurements are made and the region where position determination are to be made are large. Under this assumption spheres and cones can be considered planes to a first approximation. That is, for a sphere as large as the earth, and short distances on its surface, the earth can be considered flat. Since the approximation can yield only conservative values the only difficulty which may occur is the introduction of singularities in the ellipsoid dimensions. That is, ellipsoids of infinite dimensions may result. To obviate such difficulties it is necessary to introduce higher order approximations for specific systems at points where singularities appear. Such higher order approximations would not do for the entire system analysis because of the rule that as a mathematical model is made more realistic, it also becomes more complicated and difficult to handle. For the problem of this report, each possible position determination would require a separate calculation, whereas the first order

approximation will give a solution in closed form valid everywhere except at points where singularities appear. At such points higher order approximations can be made for meaningful results.

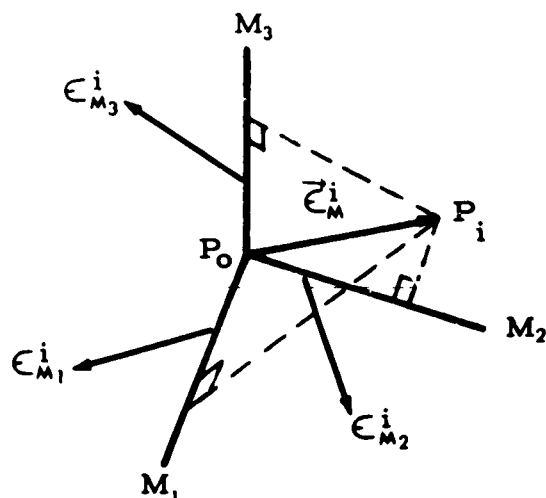


FIG. 2 THE ERROR VECTOR $\vec{\epsilon}_M^i$ IS GIVEN BY THE INTERSECTIONS OF PLANES DRAWN NORMAL TO M_1 , M_2 , M_3 FROM ERROR COMPONENTS $\epsilon_{M_1}^i$, $\epsilon_{M_2}^i$, $\epsilon_{M_3}^i$

In Fig. 2 the error vector $\vec{\epsilon}_M^i$ is found by drawing planes normal to the axes at $\epsilon_{M_1}^i$, $\epsilon_{M_2}^i$ and $\epsilon_{M_3}^i$. $\vec{\epsilon}_M^i$ gives the measured position P_i relative to the true position P_0 . It should be noted that the vector composition of Fig. 2 does not correspond to that of the ordinary vector composition of vector geometry. The method used here is necessary to correspond to physical position determination systems. The axes M_1 , M_2 , M_3 are generally non-orthogonal. Assume a new set of axes (x'_1 , x'_2 , x'_3), where x'_1 is in the plane of M_1 M_3 and normal to M_3 , x'_2 is in the plane of M_2 M_3 and normal to M_3 , and x'_3 coincides with M_3 as shown in Fig. 3.

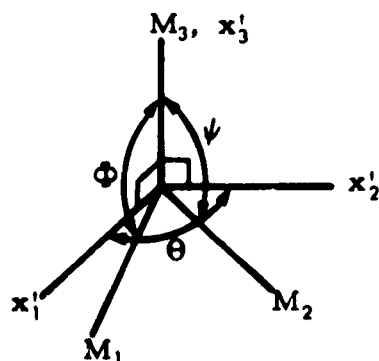


FIG. 3 A SKETCH SHOWING THE TRANSFORMATION FROM THE (M_1, M_2, M_3) SET OF COORDINATES TO THE (x'_1, x'_2, x'_3) SET

The angle between M_1 and M_3 is Φ , the angle between M_2 and M_3 is Ψ , and the angle between x'_1 and x'_2 is Θ as shown in Fig. 3. Note that the (x'_1, x'_2, x'_3) set of coordinates is also non-orthogonal since Θ is not generally 90° .

Fig. 4 shows the $M_1 M_3$ plane and the planes drawn normal to M_1 and M_3 at $\epsilon_{M_1}^i$ and $\epsilon_{M_3}^i$ intersecting at P_i' . A plane from the intersection P_i' normal to the x'_1 axis marks off $\epsilon_{x'_1}^i$ on that axis.

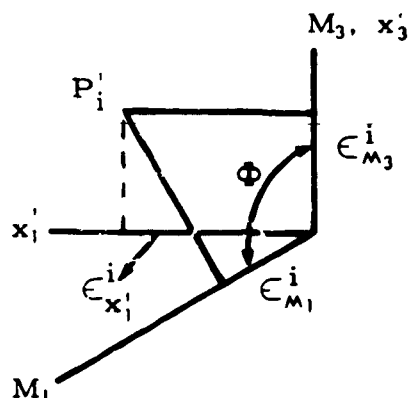


FIG. 4 THE PLANE $M_1 M_3$ IS SHOWN WITH THE PLANES DRAWN NORMAL TO M_1 AND M_3 AT $\epsilon_{M_1}^i$ AND $\epsilon_{M_3}^i$ INTERSECTING AT P_i'

From Fig. 4:

$$\epsilon_{x'_1}^i = \epsilon_{M_1}^i \csc \Phi - \epsilon_{M_3}^i \cot \Phi$$

$$\epsilon_{x'_3}^i = \epsilon_{M_3}^i$$

Fig. 5 shows the $M_2 M_3$ plane and the planes drawn normal to M_2 and M_3 at $\epsilon_{M_2}^i$ and $\epsilon_{M_3}^i$ intersecting at P_i'' . A plane from the intersection of P_i'' normal to the x'_2 axis marks off $\epsilon_{x'_2}^i$ on that axis.

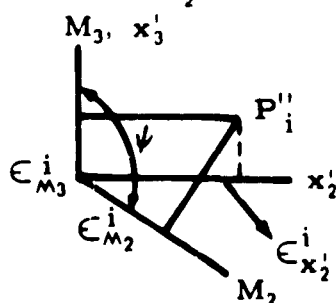


FIG. 5 THE PLANE $M_2 M_3$ IS SHOWN WITH THE PLANES DRAWN NORMAL TO M_2 AND M_3 AT $\epsilon_{M_2}^i$ AND $\epsilon_{M_3}^i$ INTERSECTING AT P_i''

From Fig. 5:

$$\epsilon_{x'_2}^i = \epsilon_{M_2}^i \csc \psi - \epsilon_{M_3}^i \cot \psi$$

$$\epsilon_{x'_3}^i = \epsilon_{M_3}^i$$

Transforming from the (M_1, M_2, M_3) coordinates to the (x'_1, x'_2, x'_3) coordinates gives:

$$\epsilon_{x'_1}^i = \epsilon_{M_1}^i \csc \Phi - \epsilon_{M_3}^i \cot \Phi$$

$$\epsilon_{x'_2}^i = \epsilon_{M_2}^i \csc \psi - \epsilon_{M_3}^i \cot \psi$$

$$\epsilon_{x'_3}^i = \epsilon_{M_3}^i$$

To transform from the (x'_1, x'_2, x'_3) coordinates to the orthogonal coordinates (x_1, x_2, x_3) consider Fig. 6. Fig. 6 shows the x'_1, x'_2 plane and the planes drawn normal to x'_1 and x'_2 at $\epsilon_{x'_1}^i$ and $\epsilon_{x'_2}^i$ intersecting at P'''_i . A plane drawn through P'''_i and normal to the x_2 axis marks off $\epsilon_{x_2}^i$ on that axis.

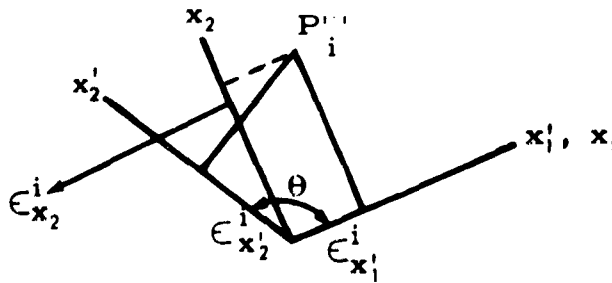


FIG. 6 THE PLANE x'_1, x'_2 IS SHOWN WITH THE PLANES DRAWN NORMAL TO x'_1 AND x'_2 AT $\epsilon_{x'_1}^i$ AND $\epsilon_{x'_2}^i$ INTERSECTING AT P'''_i

From Fig. 6:

$$\epsilon_{x_2}^i = \epsilon_{x'_2}^i \csc \theta - \epsilon_{x'_1}^i \cot \theta$$

$$\epsilon_{x_1}^i = \epsilon_{x'_1}^i$$

Therefore:

$$\begin{aligned}
\epsilon_{x_1}^i &= \epsilon_{M_1}^i \csc \Phi - \epsilon_{M_3}^i \cot \Phi \\
\epsilon_{x_2}^i &= (\epsilon_{M_2}^i \csc \psi - \epsilon_{M_3}^i \cot \psi) \csc \theta \\
&\quad - (\epsilon_{M_1}^i \csc \Phi - \epsilon_{M_3}^i \cot \Phi) \cot \theta \\
\epsilon_{x_3}^i &= \epsilon_{M_3}^i
\end{aligned}$$

In matrix notation

$$\begin{bmatrix} \epsilon_{x_1}^i \\ \epsilon_{x_2}^i \\ \epsilon_{x_3}^i \end{bmatrix} = \begin{bmatrix} \csc \Phi & 0 & -\cot \Phi \\ -\csc \Phi \cot \theta & \csc \psi \csc \theta & \cot \Phi \cot \theta - \cot \psi \csc \theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{M_1}^i \\ \epsilon_{M_2}^i \\ \epsilon_{M_3}^i \end{bmatrix}$$

The T matrix is therefore:

$$T = \begin{bmatrix} \csc \Phi & 0 & -\cot \Phi \\ -\csc \Phi \cot \theta & \csc \psi \csc \theta & \cot \Phi \cot \theta - \cot \psi \csc \theta \\ 0 & 0 & 1 \end{bmatrix}$$

and it transforms errors from a non-orthogonal coordinate set (M_1, M_2, M_3) to an arbitrary (but known) orthogonal coordinate set (x_1, x_2, x_3).

When the errors are found in an orthogonal set, they can be transformed to any other orthogonal set by using a rotation matrix R. In particular it is desirable to rotate from the (x_1, x_2, x_3) set to the principal axes set of coordinates (p_1, p_2, p_3). The rotation matrix R is not unique. The most commonly used one specifies the rotation matrix in terms of the three independent parameters called the Eulerian Angles. Since the many sources in the literature are not consistent, the R matrix will be developed in detail to show the transformations involved. The rotation R is broken up into three separate rotations. Beginning with the (x_1, x_2, x_3) set, rotate counterclockwise through an angle A about the x_3 axis, as shown in Fig. 7.

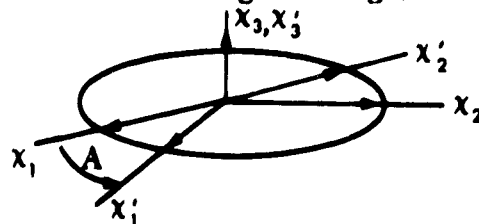


FIG. 7 SHOWS A COUNTERCLOCKWISE ROTATION ABOUT THE x_3 AXIS THROUGH THE ANGLE A

In matrix notation

$$\begin{bmatrix} \epsilon_{x'_1}^i \\ \epsilon_{x'_2}^i \\ \epsilon_{x'_3}^i \end{bmatrix} = \begin{bmatrix} \cos A & \sin A & 0 \\ -\sin A & \cos A & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{x_1}^i \\ \epsilon_{x_2}^i \\ \epsilon_{x_3}^i \end{bmatrix}$$

The rotation is given by:

$$A' = \begin{bmatrix} \cos A & \sin A & 0 \\ -\sin A & \cos A & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next rotate counterclockwise about the x'_1 axis the angle B as in Fig. 8.

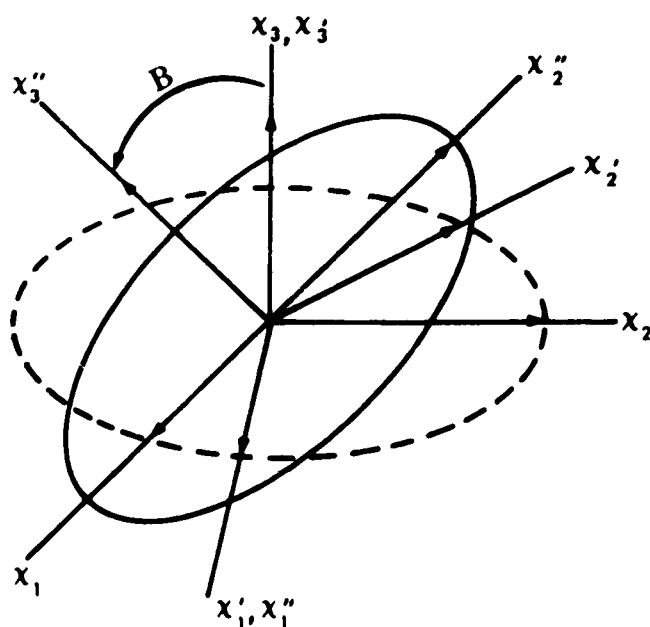


FIG. 8 SHOWS A COUNTERCLOCKWISE ROTATION ABOUT THE x'_1 AXIS THROUGH THE ANGLE B

$$\begin{bmatrix} \epsilon_{x''_1}^i \\ \epsilon_{x''_2}^i \\ \epsilon_{x''_3}^i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos B & \sin B \\ 0 & -\sin B & \cos B \end{bmatrix} \begin{bmatrix} \epsilon_{x'_1}^i \\ \epsilon_{x'_2}^i \\ \epsilon_{x'_3}^i \end{bmatrix}$$

The rotation is given by:

$$B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos B & \sin B \\ 0 & -\sin B & \cos B \end{bmatrix}$$

Finally rotate the (x_1'', x_2'', x_3'') set the angle C counterclockwise about the x_3'' axis as in Fig. 9.

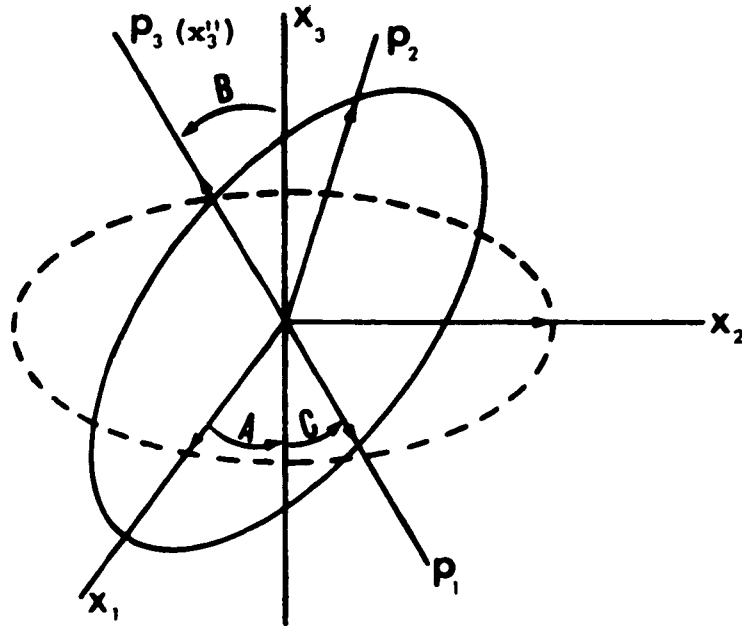


FIG. 9 SHOWS A COUNTERCLOCKWISE ROTATION ABOUT THE x_3'' AXIS THROUGH THE ANGLE C

$$\begin{bmatrix} \epsilon_{p_1}^i \\ \epsilon_{p_2}^i \\ \epsilon_{p_3}^i \end{bmatrix} = \begin{bmatrix} \cos C & \sin C & 0 \\ -\sin C & \cos C & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{x_1''}^i \\ \epsilon_{x_2''}^i \\ \epsilon_{x_3''}^i \end{bmatrix}$$

The rotation is given by:

$$C' = \begin{bmatrix} \cos C & \sin C & 0 \\ -\sin C & \cos C & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because the three-dimensional rotation matrices form a non-abelian group the order of rotation is important [7]. In general:

$$A' B' \neq B' A'$$

The distributive law holds, however, so that:

$$(A' B') (C') = (A') (B' C') = A' B' C'$$

The rotation matrix R is, therefore:

$$R = A' B' C'$$

Multiplying out yields:

$$R = \begin{bmatrix} \cos C \cos A & \cos C \sin A & \sin C \sin B \\ -\cos B \sin A \sin C & + \cos B \cos A \sin C & \\ -\sin C \cos A & -\sin C \sin A & \cos C \sin B \\ -\cos B \sin A \cos C & + \cos B \cos A \cos C & \\ \sin B \sin A & -\sin B \cos A & \cos B \end{bmatrix}$$

so that

$$\begin{bmatrix} \epsilon_{p_1}^i \\ \epsilon_{p_2}^i \\ \epsilon_{p_3}^i \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} \epsilon_{x_1}^i \\ \epsilon_{x_2}^i \\ \epsilon_{x_3}^i \end{bmatrix}$$

and

$$\begin{bmatrix} \epsilon_{p_1}^i \\ \epsilon_{p_2}^i \\ \epsilon_{p_3}^i \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} \epsilon_{m_1}^i \\ \epsilon_{m_2}^i \\ \epsilon_{m_3}^i \end{bmatrix}$$

The components of error in the principal axes coordinate set, therefore, can be determined using R and T. The variances in the principal axes set are defined by:

$$\sigma_{p_j}^2 = \frac{1}{N} \sum_{i=1}^N (\epsilon_{p_j}^i)^2 \quad (j = 1, 2, 3)$$

The variances are found as functions of σ_{M_1} , σ_{M_2} , σ_{M_3} , Φ , ψ , θ , A , B , C . σ_{M_1} , σ_{M_2} , σ_{M_3} are given. Φ , ψ , θ are determined by the geometry of the particular physical system under consideration. A , B , C are determined from:

$$\rho_{jk} = \frac{1}{N \sigma_{p_j} \sigma_{p_k}} \sum_{i=1}^N \epsilon_{p_j}^i \epsilon_{p_k}^i = 0 \quad \begin{cases} j, k = 1, 2, 3 \\ j \neq k \end{cases}$$

The σ_{p_j} are defined completely and the physical dimensions of the error density ellipsoid are known. The spatial orientation of the ellipsoid is also known in terms of the angles A , B , and C . The probability that a position determination falls into a one, two, or three σ ellipsoid is .19874, .73854, and .97072, respectively.

There is an alternative method for solving for the σ_{p_j} using formal matrix methods given in some texts [8] which will be shown here in order to derive a useful approximation. Rather than finding error components in the (p_1, p_2, p_3) axes, the error components are determined in the orthogonal set (x_1, x_2, x_3) by using the T matrix. Then the so-called covariance matrix Λ_x is developed where:

$$\Lambda_x = \begin{bmatrix} \sigma_{x_1}^2 & \rho_{12} \sigma_{x_1} \sigma_{x_2} & \rho_{13} \sigma_{x_1} \sigma_{x_3} \\ \rho_{21} \sigma_{x_2} \sigma_{x_1} & \sigma_{x_2}^2 & \rho_{23} \sigma_{x_2} \sigma_{x_3} \\ \rho_{31} \sigma_{x_3} \sigma_{x_1} & \rho_{32} \sigma_{x_3} \sigma_{x_2} & \sigma_{x_3}^2 \end{bmatrix}$$

by using the definitions:

$$\rho_{jk} = \frac{1}{N \sigma_{x_j} \sigma_{x_k}} \sum_{i=1}^N \epsilon_{x_j}^i \epsilon_{x_k}^i \quad \begin{cases} j, k = 1, 2, 3 \\ j \neq k \end{cases}$$

$$\rho_{jj} = 1 \quad (j = 1, 2, 3)$$

The formal matrix methodology for diagonalizing the covariance matrix is the similarity transformation [9]:

$$\Lambda_p = R \Lambda_x R^T$$

where R^T is the transposed R matrix and where:

$$\Lambda_p = \begin{bmatrix} \sigma_{p_1}^2 & \rho_{12} \sigma_{p_1} \sigma_{p_2} & \rho_{13} \sigma_{p_1} \sigma_{p_3} \\ \rho_{21} \sigma_{p_2} \sigma_{p_1} & \sigma_{p_2}^2 & \rho_{23} \sigma_{p_2} \sigma_{p_3} \\ \rho_{31} \sigma_{p_3} \sigma_{p_1} & \rho_{32} \sigma_{p_3} \sigma_{p_2} & \sigma_{p_3}^2 \end{bmatrix}$$

Since the off-diagonal elements of a diagonal matrix vanish; that is

$$\rho_{jk} \sigma_{p_j} \sigma_{p_k} = \frac{1}{N} \sum_{i=1}^N \epsilon_{p_j}^i \epsilon_{p_k}^i = 0 \quad \begin{cases} j, k = 1, 2, 3 \\ j \neq k \end{cases}$$

the angles A, B, and C of the R matrix can be determined. The σ_{p_i} , therefore, are determined and the ellipsoid orientation is also found from A, B, and C. It is evident that the probability of finding a position determination in a sphere containing an ellipsoid is greater than the probability of finding it in the ellipsoid. For example, the probability of finding a position determination in a sphere containing a one σ ellipsoid is greater than 0.19874. One such sphere whose radius is always larger than any σ_{p_j} is one whose radius is:

$$d_{rms} = (\sigma_{p_1}^2 + \sigma_{p_2}^2 + \sigma_{p_3}^2)^{1/2}$$

where $(\sigma_{p_1}^2 + \sigma_{p_2}^2 + \sigma_{p_3}^2)$ is the trace of the covariance matrix Λ_p . The trace of an orthogonal matrix is invariant under a similarity transformation [10], therefore,

$$d_{rms} = (\sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_3}^2)^{1/2}$$

It is possible, under some conditions, to get a conservative estimate of a system without using the R matrix which will greatly simplify calculations. For a system where the probability of finding a position determination must be large (say 0.95 or more) the d_{rms} statistic can be a meaningful tool in providing a figure-of-merit. For example, assume that a system must provide a probability of 0.97 that a position determination is within 10 miles of the true position for some given region of space. It is clear that a system whose three σ ellipsoid is contained in a sphere whose radius

$$D(0.97) = 3 d_{rms} = \left[(3\sigma_{x_1})^2 + (3\sigma_{x_2})^2 + (3\sigma_{x_3})^2 \right]^{1/2}$$

is less than 10 miles, is a useful system. Use of the d_{rms} statistic is made whenever system considerations allow (see reference [11]).

APPLICATIONS

1. RHO-RHO-RHO Satellite System. The system to be analyzed will consist of two satellites used as follows: A ground station would transmit a radio signal to a satellite which would relay it to an aircraft. The aircraft would receive and retransmit the signal and the aircraft altitude back to the original satellite and also to a second satellite. They,

in turn, would relay the signals back to the ground station. Knowledge of the satellite positions and the total delay times involved in sending and receiving the signal is used to determine the distances between aircraft and satellites. From this information and knowledge of aircraft altitude, a position can be determined [3] and [4].

The three ranges from which a position determination can be made are the two distances between the aircraft and two satellites and the distance from the center of the earth to the aircraft, which must be found from aircraft altitude. Because the earth is not a perfect sphere, an iterative computational procedure must be followed to get an accurate absolute fix on aircraft position. When a position determination is to be made, a computation is made using the two satellite-aircraft ranges and the aircraft altitude added to some average value of earth radius. The approximate position is thus determined and a new value of earth radius for the approximate position is found. Another computation is made using the two satellite-aircraft ranges and the aircraft altitude added to the new value of earth radius. An accurate position determination is thus made. If necessary, a number of iterations can be made refining the position determination until an accurate reading is found. In practice, it would be quite simple to determine the accuracy of a reading simply by examining the result of two or three iterations of the computation. When the difference between results is negligible, the result is accurate.

In order to properly use the probabilistic system model developed in the previous part of this report, it is necessary to identify the Φ , ψ , θ angles of the T matrix with a set of system angles.

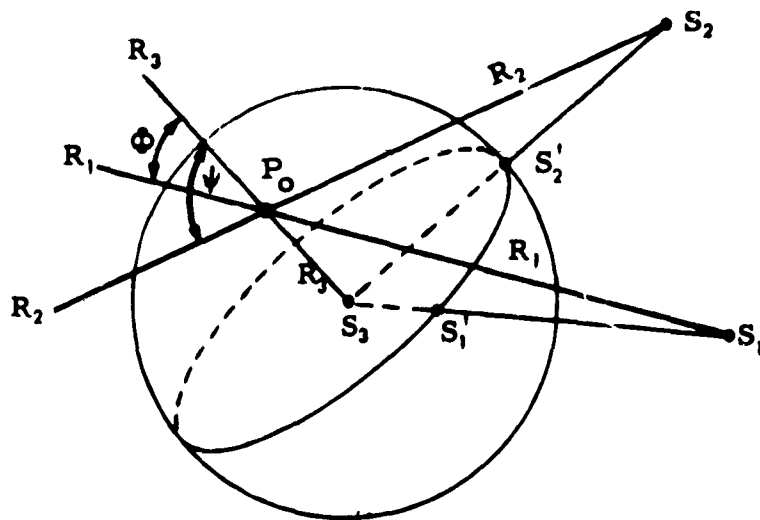


FIG. 10 A TWO-SATELLITE SYSTEM SHOWING THE SYSTEM ANGLES IDENTIFIED AS THE Φ AND ψ ANGLES OF THE T MATRIX OF THE PROBABILISTIC SYSTEM MODEL

Fig. 10 is a sketch of a two-satellite RHO-RHO-RHO system showing the system angles which are chosen to correspond to the T matrix Φ , ψ angles of the probabilistic system model. They are the angles between the R_1 , R_3 and the R_2 , R_3 axes, respectively. Fig. 10 also shows the subsatellite points S'_1 and S'_2 which are the points of intersection of the earth sphere and the lines $\overline{S_1 S_3}$ and $\overline{S_2 S_3}$, respectively. The θ angle of the T matrix corresponds to the satellite system angle formed by the tangents of the great circle arcs drawn between the subsatellite point S'_1 and P_0 and the subsatellite point S'_2 and P_0 as in Fig. 11.

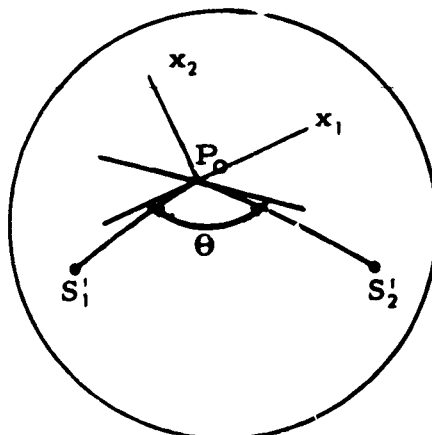


FIG. 11 THE θ ANGLE OF THE T MATRIX CORRESPONDS TO THE SYSTEM ANGLE BETWEEN THE TANGENTS OF THE GREAT CIRCLE ARCS DRAWN BETWEEN S'_1 AND P_0 AND BETWEEN S'_2 AND P_0

Fig. 11 also shows the x_1 , x_2 axes of the orthogonal coordinates (x_1 , x_2 , x_3) into which errors will be transformed by the T matrix. That is,

$$\begin{bmatrix} \epsilon_{x_1}^i \\ \epsilon_{x_2}^i \\ \epsilon_{x_3}^i \end{bmatrix} = \begin{bmatrix} \csc \Phi & 0 & -\cot \Phi \\ -\csc \Phi \cot \theta & \csc \psi \csc \theta & \cot \Phi \cot \theta - \cot \psi \csc \theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{R_1}^i \\ \epsilon_{R_2}^i \\ \epsilon_{R_3}^i \end{bmatrix}$$

where R_3 and x_3 are coincident and normal to the earth sphere at P_0 and where x_1 and x_2 are orthogonal to each other and to x_3 and in a plane tangent to the earth at P_0 with x_1 coincident with the tangent of the great circle arc $\overline{S'_1 P_0}$ at P_0 . The errors in the (x_1 , x_2 , x_3) coordinate set are therefore:

$$\begin{aligned} \epsilon_{x_1}^i &= \epsilon_{R_1}^i \csc \Phi - \epsilon_{R_3}^i \cot \Phi \\ \epsilon_{x_2}^i &= -\epsilon_{R_1}^i \csc \Phi \cot \theta + \epsilon_{R_2}^i \csc \psi \csc \theta \end{aligned}$$

$$+ \epsilon_{R_3}^i \cot \Phi \cot \Theta - \epsilon_{R_3}^i \cot \psi \csc \Theta$$

$$\epsilon_{x_3}^i = \epsilon_{R_3}^i$$

Under the assumption that for air traffic control interest is in the position of an aircraft in the plane tangent to the earth at P_0 , it is necessary to consider only the errors in such tangent plane. For the two-satellite system, therefore, only $\epsilon_{x_1}^i$ and $\epsilon_{x_2}^i$, the errors along the orthogonal axes in the tangent plane need to be considered. Since the ellipsoid is a surface, all plane sections of which are ellipses, the loci of constant probability error density in the plane defined by the coordinates x_1 and x_2 are ellipses. They are called bivariate normal equi-probability error density ellipses. The two-dimensional probability density function for a normal distribution in the (x_1, x_2) coordinate set is:

$$p(\epsilon_{x_1}, \epsilon_{x_2}) = \frac{e^{-\frac{1}{2} Q(\epsilon_{x_1}, \epsilon_{x_2})}}{2\pi \sigma_{x_1} \sigma_{x_2} \sqrt{|e|}}$$

$$\text{where } Q(\epsilon_{x_1}, \epsilon_{x_2}) = \frac{1}{|e|} \sum_{i,k} e_{ik} \frac{\epsilon_{x_i} \epsilon_{x_k}}{\sigma_{x_i} \sigma_{x_k}} \quad (i, k = 1, 2)$$

$$e = \begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix}$$

$$e_{jk} = \frac{1}{\sigma_{x_j} \sigma_{x_k} N} \sum_{i=1}^N \epsilon_{x_j}^i \epsilon_{x_k}^i \quad \begin{cases} j, k = 1, 2 \\ j = k \end{cases}$$

$$e_{jj} = 1 \quad (j = 1, 2)$$

If the errors were known in the two-dimensional principal axes coordinate set (p_1, p_2) then:

$$e_{jk} = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases} \quad (j, k = 1, 2)$$

Therefore,

$$|e| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$Q = \frac{\epsilon_{p_1}^2}{\sigma_{p_1}^2} + \frac{\epsilon_{p_2}^2}{\sigma_{p_2}^2}$$

$$\text{where } \sigma_{p_j}^2 = \frac{1}{N} \sum_{i=1}^N \left(\epsilon_{p_j}^i \right)^2 \quad (j = 1, 2)$$

The probability density function in the principal axes set is:

$$p(\epsilon_{p_1}, \epsilon_{p_2}) = \frac{e^{-\frac{1}{2} \left(\frac{\epsilon_{p_1}^2}{\sigma_{p_1}^2} + \frac{\epsilon_{p_2}^2}{\sigma_{p_2}^2} \right)}}{2\pi \sigma_{p_1}^2 \sigma_{p_2}^2}$$

To find the probability, P_c , that a position determination is in an ellipse whose axes are $c\sigma_{p_1}$ and $c\sigma_{p_2}$ where c is a constant, evaluate the integral

$$P_c = \frac{1}{2\pi} \iint \frac{e^{-\frac{1}{2} \left(\frac{\epsilon_{p_1}^2}{\sigma_{p_1}^2} + \frac{\epsilon_{p_2}^2}{\sigma_{p_2}^2} \right)} d\epsilon_{p_1} d\epsilon_{p_2}}{\sigma_{p_1} \sigma_{p_2}}$$

$$\text{let } \frac{\epsilon_{p_1}}{\sigma_{p_1}} = z_1 \quad dz_1 = \frac{d\epsilon_{p_1}}{\sigma_{p_1}}$$

$$\frac{\epsilon_{p_2}}{\sigma_{p_2}} = z_2 \quad dz_2 = \frac{d\epsilon_{p_2}}{\sigma_{p_2}}$$

$$P_c = \frac{1}{2\pi} \iint e^{-\frac{1}{2} (z_1^2 + z_2^2)} dz_1 dz_2$$

in polar coordinates

$$z_1 = r \cos \theta$$

$$z_2 = r \sin \theta$$

$$z_1^2 + z_2^2 = r^2$$

$$dz_1 dz_2 = r dr d\theta$$

$$P_c = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^c e^{-\frac{r^2}{2}} r dr$$

$$\text{where } c^2 = \frac{\epsilon_{p_1}^2}{\sigma_{p_1}^2} + \frac{\epsilon_{p_2}^2}{\sigma_{p_2}^2}$$

$$P_c = \int_0^c e^{-\frac{r^2}{2}} r dr$$

$$\text{let } y = \frac{r^2}{2}$$

$$dy = r dr$$

$$P_c = \int_0^{\frac{c^2}{2}} e^{-y} dy$$

$$P_c = 1 - e^{-\frac{c^2}{2}}$$

$$P_c = 1 - \sqrt{2\pi} \left(\frac{e^{-\frac{c^2}{2}}}{\sqrt{2\pi}} \right)$$

The quantity in brackets can be found in tables of Probability and Statistics books. A short table of P_c values is given in Table II.

TABLE II

THE PROBABILITIES, P_c , THAT A POSITION DETERMINATION FALLS IN A ONE, TWO, TWO AND ONE-HALF OR THREE SIGMA ELLIPSE

c	P_c
1	.3935
2	.8647
2.5	.9561
3	.9889

Table II shows that the probability that a position determination falls into a one, two, two and one-half, and three sigma ellipse is .3935, .8647, .9561 and .9889, respectively. The physical axes can be found by using the R matrix of the system model developed in the previous part of this report. To simplify computations assume that a satellite data acquisition system should provide a minimum probability of .95 that a position determination is within some specified distance from the true position. A circle which contains the 2.5 sigma ellipse (.9561 probability) is one whose radius is $D = 2.5 \text{ drms}$ where

$$\text{drms} = \left[\sigma_{x_1}^2 + \sigma_{x_2}^2 \right]^{\frac{1}{2}}$$

where

$$\sigma_{x_1}^2 = \frac{1}{N} \sum_{i=1}^N (\epsilon_{x_1}^i)^2$$

$$\sigma_{x_2}^2 = \frac{1}{N} \sum_{i=1}^N (\epsilon_{x_2}^i)^2$$

$$\epsilon_{x_1}^i = \epsilon_{R_1}^i \csc \Phi - \epsilon_{R_3}^i \cot \Phi$$

$$\begin{aligned} \epsilon_{x_2}^i = & -\epsilon_{R_1}^i \csc \Phi \cot \Theta + \epsilon_{R_2}^i \csc \psi \csc \Theta \\ & + \epsilon_{R_3}^i \cot \Phi \cot \Theta - \epsilon_{R_3}^i \cot \psi \csc \Theta \end{aligned}$$

$$\sigma_{x_1}^2 = \frac{1}{N} \sum_{i=1}^N (\epsilon_{R_1}^i \csc \Phi - \epsilon_{R_3}^i \cot \Phi)^2$$

$$\sigma_{x_1}^2 = \sigma_{R_1}^2 \csc^2 \Phi + \sigma_{R_3}^2 \cot^2 \Phi$$

since $\epsilon_{R_1}^i, \epsilon_{R_3}^i$ are independent random variables, i.e.,

$$\frac{1}{N} \sum_{i=1}^N -\epsilon_{R_1}^i \epsilon_{R_3}^i \csc \Phi \cot \Phi = 0$$

Similarly,

$$\sigma_{x_2}^2 = \sigma_{R_1}^2 \csc^2 \Phi \cot^2 \Theta + \sigma_{R_2}^2 \csc^2 \psi \csc^2 \Theta + \sigma_{R_3}^2 (\cot \Phi \cot \Theta - \cot \psi \csc \Theta)^2$$

$$D = 2.5 \, d_{rms}$$

$$\begin{aligned} D = 2.5 \left[& \sigma_{R_1}^2 \csc^2 \Phi + \sigma_{R_3}^2 \cot^2 \Phi \right. \\ & + \sigma_{R_1}^2 \csc^2 \Phi \cot^2 \Theta + \sigma_{R_2}^2 \csc^2 \psi \csc^2 \Theta \\ & + \sigma_{R_3}^2 \cot^2 \Phi \cot^2 \Theta + \sigma_{R_3}^2 \cot^2 \psi \csc^2 \Theta \\ & \left. - 2 \sigma_{R_3}^2 \cot \Phi \cot \Theta \cot \psi \csc \Theta \right]^{1/2} \end{aligned}$$

$$\begin{aligned} D = 2.5 \left[& \sigma_{R_1}^2 \csc^2 \Phi \csc^2 \Theta \right. \\ & + \sigma_{R_2}^2 \csc^2 \psi \csc^2 \Theta \\ & \left. + \sigma_{R_3}^2 (\cot^2 \Phi \csc^2 \Theta + \cot^2 \psi \csc^2 \Theta - 2 \cot \Phi \cot \psi \cot \Theta \csc \Theta) \right]^{1/2} \end{aligned}$$

Assuming that there will be some error in the satellite position determinations and possibly in the measurement of earth radius $\sigma_{R_1}^2$, $\sigma_{R_2}^2$ and $\sigma_{R_3}^2$ are replaced by $\sigma_{R_{1T}}^2$, $\sigma_{R_{2T}}^2$ and $\sigma_{R_{3T}}^2$, respectively, where:

$$\sigma_{R_{1T}}^2 = \sigma_{R_1}^2 + \sigma_{S_1}^2$$

$$\sigma_{R_{2T}}^2 = \sigma_{R_2}^2 + \sigma_{S_2}^2$$

$$\sigma_{R_{3T}}^2 = \sigma_{R_3}^2 + \sigma_{S_3}^2$$

and σ_{S_1} , σ_{S_2} , σ_{S_3} are the standard deviations derived from errors in the measurements of position of the stations S_1 , S_2 and S_3 , respectively.

To find $\epsilon_{S_1}^i$, $\epsilon_{S_2}^i$, $\epsilon_{S_3}^i$ consider Fig. 12:

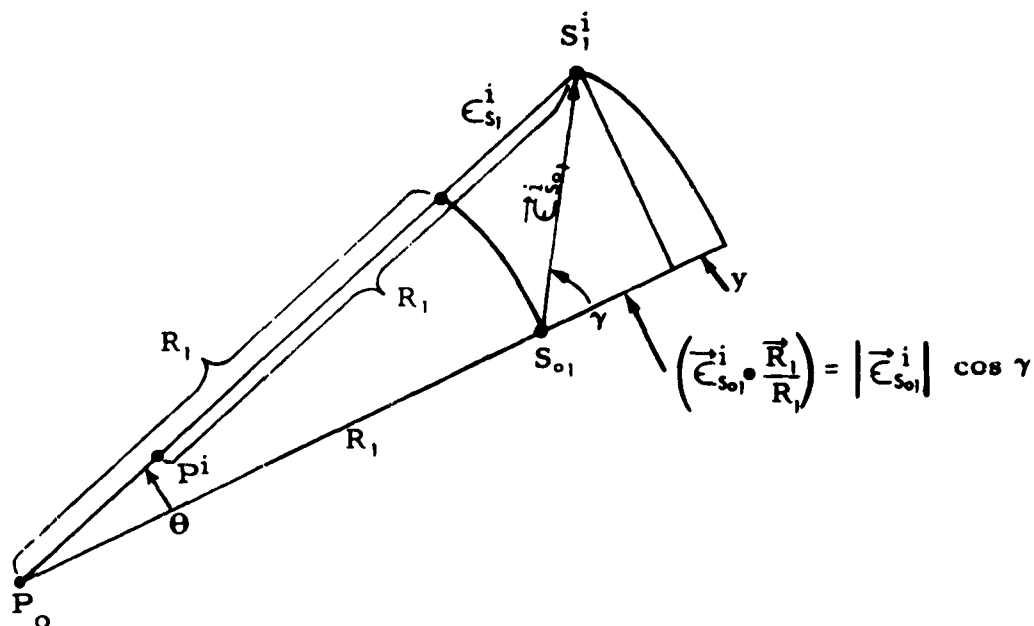


FIG. 12 P_0 IS THE TRUE POSITION, P^i THE MEASURED POSITION DUE TO ERROR IN MEASURING THE STATION POSITION ALONE

From Fig. 12:

$$\epsilon_{S_1}^i = |\vec{\epsilon}_{S_0^i}^i| \cos \gamma + y$$

$$y = (R_1 + |\vec{\epsilon}_{S_0^i}^i| \cos \gamma) (1 - \cos \theta)$$

Therefore,

$$\epsilon_{S_1}^i = |\vec{\epsilon}_{S_0^i}^i| \cos \gamma + (R_1 + |\vec{\epsilon}_{S_0^i}^i| \cos \gamma) (1 - \cos \theta)$$

where $\vec{e}_{s_{0i}}^i$ is the i'th error vector to be determined for the particular system used to measure the position of station one. For small θ , $\epsilon_{s_1}^i \rightarrow |\vec{e}_{s_{01}}^i| \cos \gamma$, which is the component of the error vector on R_1 . For the satellite system under consideration, where the ranges are thousands of miles and the errors are of the order of a few miles (a conservative estimate), θ is exceedingly small and $\epsilon_{s_1}^i, \epsilon_{s_2}^i, \epsilon_{s_3}^i$ can be taken as the components of the error vectors $\vec{e}_{s_{01}}^i, \vec{e}_{s_{02}}^i, \vec{e}_{s_{03}}^i$ along R_1, R_2 , and R_3 , respectively. The errors in the i'th measurements on the ranges are:

$$\epsilon_{R_{1T}}^i = \epsilon_{R_1}^i + \epsilon_{s_1}^i$$

$$\epsilon_{R_{2T}}^i = \epsilon_{R_2}^i + \epsilon_{s_2}^i$$

$$\epsilon_{R_{3T}}^i = \epsilon_{R_3}^i + \epsilon_{s_3}^i$$

Therefore:

$$\sigma_{R_{1T}}^2 = \sigma_{R_1}^2 + \sigma_{s_1}^2$$

$$\sigma_{R_{2T}}^2 = \sigma_{R_2}^2 + \sigma_{s_2}^2$$

$$\sigma_{R_{3T}}^2 = \sigma_{R_3}^2 + \sigma_{s_3}^2$$

Since the correlations

$$\rho_{R_j S_j} = \frac{1}{N \sigma_{R_j} \sigma_{s_j}} \sum_{i=1}^N \epsilon_{R_j}^i \epsilon_{s_j}^i = 0 \quad (j = 1, 2, 3)$$

Therefore:

$$D = 2.5 \left[\sigma_{R_{1T}}^2 \csc^2 \Phi \csc^2 \theta + \sigma_{R_{2T}}^2 \csc^2 \psi \csc^2 \theta + \sigma_{R_{3T}}^2 (\cot^2 \Phi \csc^2 \theta + \cot^2 \psi \csc^2 \theta - 2 \cot \Phi \cot \psi \cot \theta \csc \theta) \right]^{\frac{1}{2}}$$

In order to get a mapping of error isograms (constant D contours) on the earth it is necessary to find functional relationships between the Φ , ψ , and θ parameters to some set of earth-related parameters.

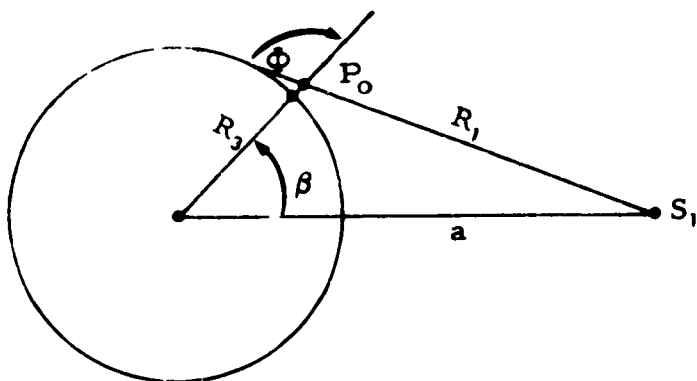


FIG. 13 THE SYSTEM ANGLE Φ IS RELATED TO THE EARTH ANGLE β

From Fig. 13:

$$a^2 = R_1^2 + R_3^2 - 2 R_1 R_3 \cos \Phi$$

$$R_1^2 = a^2 + R_3^2 - 2 a R_3 \cos \beta$$

Eliminating R_1 and solving for Φ gives:

$$\cos \Phi = \frac{a \cos \beta - R_3}{(a^2 + R_3^2 - 2 a R_3 \cos \beta)^{1/2}} \quad (6)$$

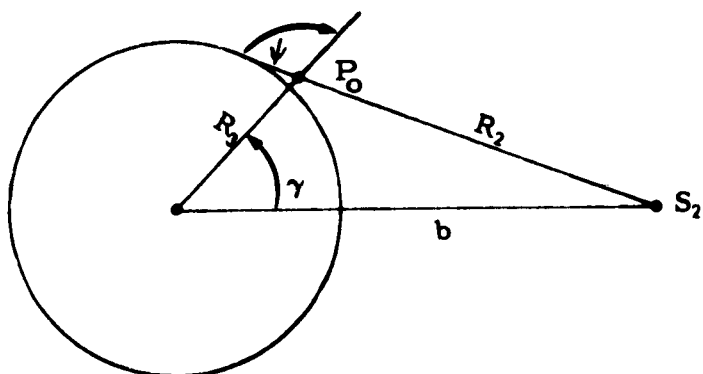


FIG. 14 THE SYSTEM ANGLE ψ IS RELATED TO THE EARTH ANGLE γ

From Fig. 14:

$$b^2 = R_2^2 + R_3^2 - 2 R_2 R_3 \cos \psi$$

$$R_2^2 = b^2 + R_3^2 - 2 b R_3 \cos \gamma$$

Eliminating R_2 and solving for ψ gives:

$$\cos \psi = \frac{b \cos \gamma - R_3}{(b^2 + R_3^2 - 2 b R_3 \cos \gamma)} \quad (7)$$

With the help of Fig. 15 it is possible to transform from β, γ, θ to L, λ_1, λ_2 a set of earth angles which become Latitude and Longitude angles when the two satellite stations are directly over the equator.

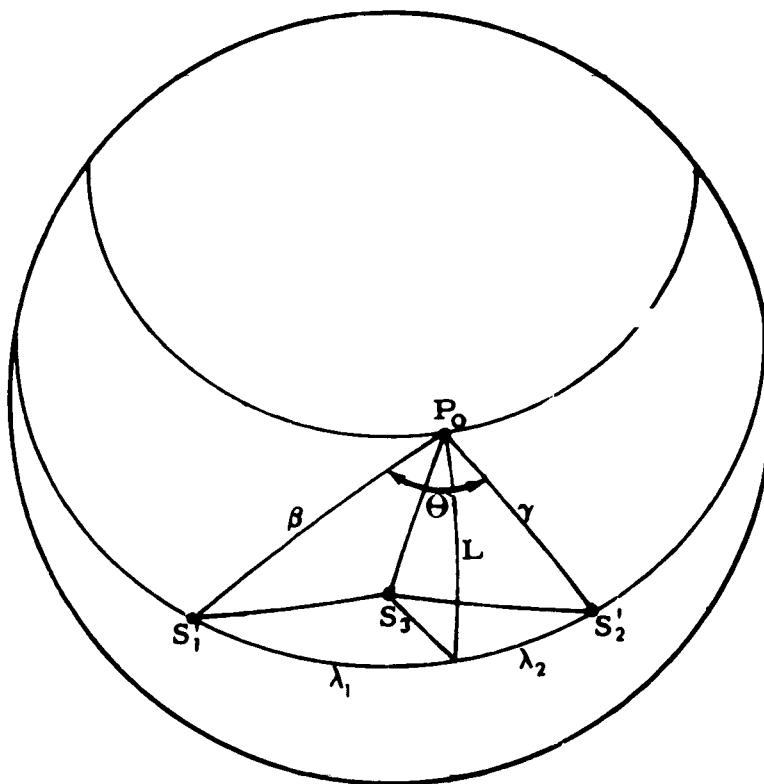


FIG. 15 THE ANGLES β, γ RELATED TO THE L, λ_1, λ_2 COORDINATE SET

From Fig. 15 using spherical trigonometry:

$$\cos (\lambda_1 + \lambda_2) = \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos \theta$$

$$\cos \beta = \cos L \cos \lambda_1$$

$$\cos \gamma = \cos L \cos \lambda_2$$

Using equations (6) and (7) and the equations derived from Fig. 15:

$$\cos \theta = \frac{\cos (\lambda_1 + \lambda_2) - \cos L \cos \lambda_1 \cos L \cos \lambda_2}{[(1 - \cos^2 L \cos^2 \lambda_1)(1 - \cos^2 L \cos^2 \lambda_2)]^{1/2}}$$

$$\cos \Phi = \frac{a \cos L \cos \lambda_1 - R_3}{(a^2 + R_3^2 - 2 a R_3 \cos L \cos \lambda_1)^{1/2}}$$

$$\cos \psi = \frac{b \cos L \cos \lambda_2 - R_3}{(b^2 + R_3^2 - 2 b R_3 \cos L \cos \lambda_2)^{1/2}}$$

Summarizing:

$$D = 2.5 \left[\sigma_{R_1}^2 \csc^2 \Phi \csc^2 \theta + \sigma_{R_2}^2 \csc^2 \psi \csc^2 \theta + \sigma_{R_3}^2 (\cot^2 \Phi \csc^2 \theta + \cot^2 \psi \csc^2 \theta - 2 \cot \Phi \cot \psi \cot \theta \csc \theta) \right]^{1/2}$$

where:

$$\cos \theta = \frac{\cos (\lambda_1 + \lambda_2) - \cos^2 L \cos \lambda_2 \cos \lambda_1}{(1 - \cos^2 L \cos^2 \lambda_1)(1 - \cos^2 L \cos^2 \lambda_2)^{1/2}}$$

$$\cos \Phi = \frac{a \cos L \cos \lambda_1 - R_3}{(a^2 + R_3^2 - 2 a R_3 \cos L \cos \lambda_1)^{1/2}}$$

$$\cos \psi = \frac{b \cos L \cos \lambda_2 - R_3}{(b^2 + R_3^2 - 2 b R_3 \cos L \cos \lambda_2)^{1/2}}$$

where a and b are the distances between the earth center and the satellites S_1 and S_2 , respectively. A computer program was written for a two-satellite system where the satellites were spaced at angles ranging from 40 to 70 degrees on the equator in synchronous altitude orbits. To map constant D values (error isograms), it is necessary to make an estimate on the σ_{R_1} , σ_{R_2} , σ_{R_3} and σ_{S_1} , σ_{S_2} , σ_{S_3} . References [3], [4] and [5] include detailed analyses and studies from which the values can be estimated. It is assumed that the satellite ranges can be measured to within 300 feet (one-sigma value) and the aircraft altitude can likewise be measured to within 300 feet (one-sigma value). Since more sophisticated equipment can be used to measure the satellite positions, it is assumed that σ_{S_1} , σ_{S_2} are 100 feet (one-sigma value). σ_{S_3} , the measure of the position of the earth center is assumed known to such a close value that it is taken as zero. That is, it is assumed that:

$$\sigma_{R_1} = \sigma_{R_2} = \sigma_{R_3} = 300 \text{ feet}$$

$$\sigma_{S_1} = \sigma_{S_2} = 100 \text{ feet}$$

$$\sigma_{S_3} = 0$$

Using these values, mappings of constant D values in nautical miles are shown for the North Atlantic Ocean Area in Figs. 17 to 20 for angular satellite spacings of 40, 50, 60, and 70 degrees. Because the values of assumed standard deviations are rather optimistic and are based upon an attempt to minimize them, the set of mappings in Figs. 21 to 24 are based upon a more conservative set of assumed standard deviations as follows:

$$\sigma_{R_1} = \sigma_{R_2} = \sigma_{R_3} = 900 \text{ feet}$$

$$\sigma_{S_1} = \sigma_{S_2} = 300 \text{ feet}$$

$$\sigma_{S_3} = 0$$

When a position determination is made, the probability is greater than .95 that it is within D nautical miles of the true position.

Figs. 17 to 24 show that excellent results are attainable for the two-satellite system over a large portion of the North Atlantic Ocean including the heavily travelled air route between Newfoundland and Ireland. Far north regions are excluded and large D values are indicated near the equator. The equation for D, in fact, indicates that at a position on the equator, D is infinite. The reading is physically absurd and is due to the first order approximation used in constructing the probabilistic model in the General Analysis. In finding the error vector for a particular measurement it was assumed that for large spheres, small portions of spherical surfaces could be considered planes. The assumption leads to the singularity in D at the equator. To get an upper bound on the error that can be expected at the equator, it is necessary to apply a second order approximation. It is apparent from geometrical considerations that the subsatellite points S_1' and S_2' shown in Fig. 16 will be measured with the least precision.

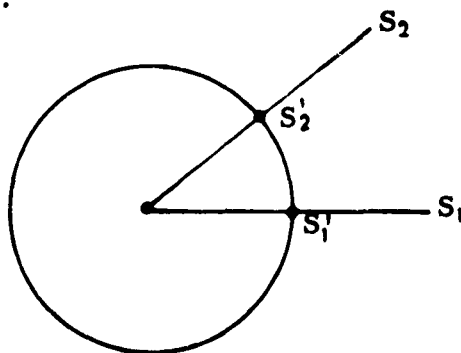
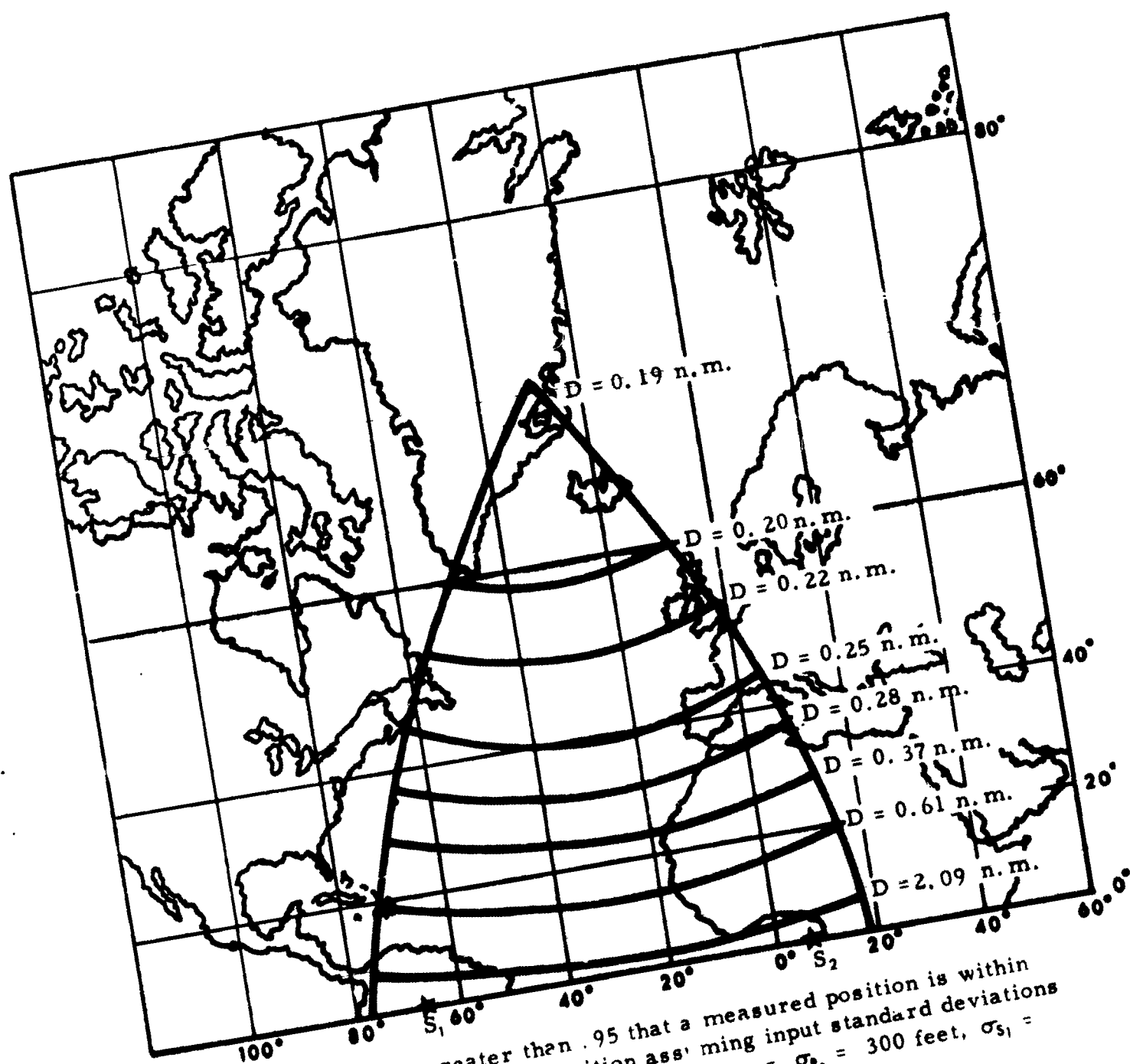
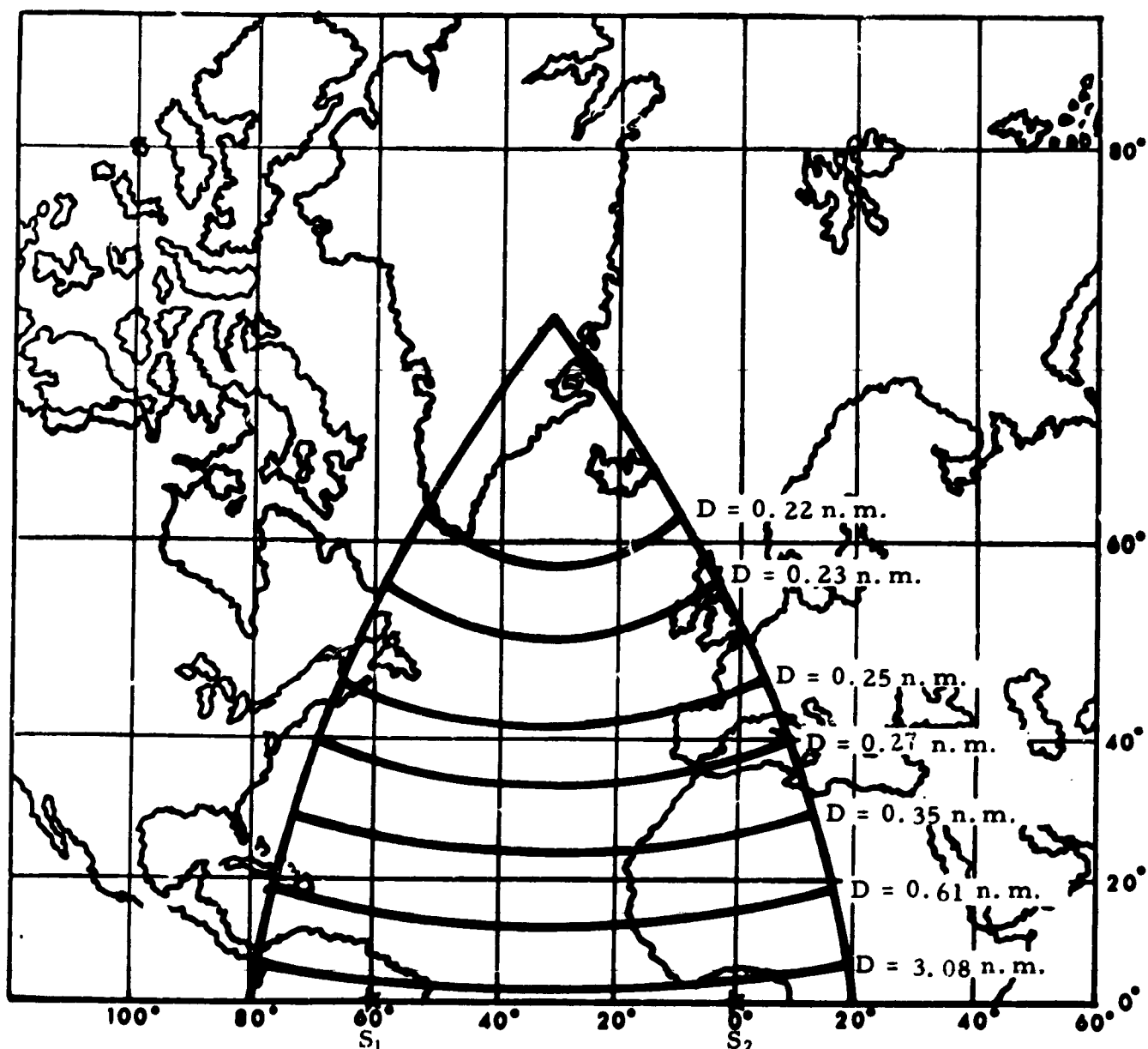


FIG. 16 THE SUBSATELLITE POINTS S_1' , S_2' ON THE EQUATOR ARE MEASURED WITH THE LEAST PRECISION



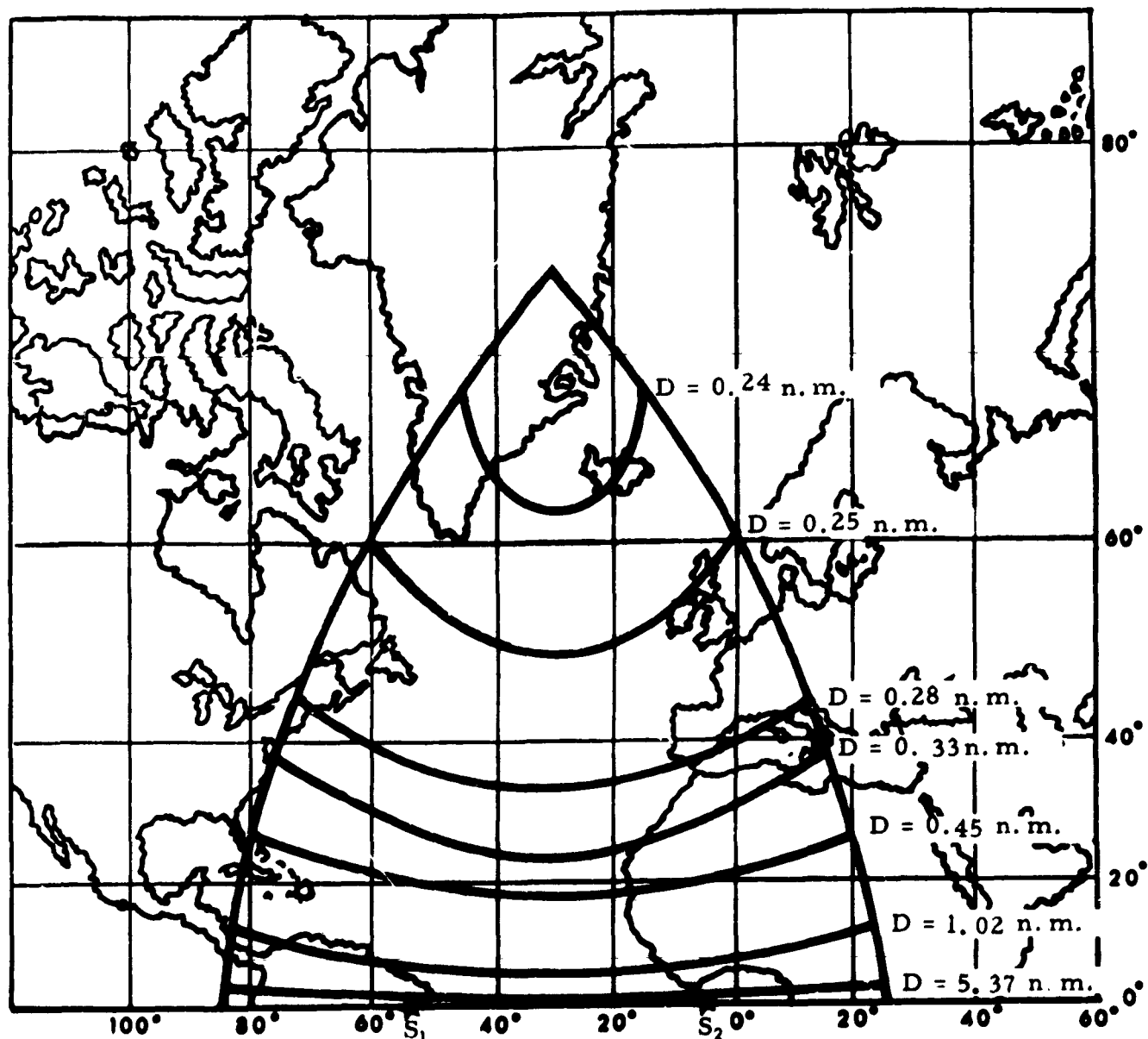
The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming input standard deviations as follows (see text for definitions): $\sigma_{r_1} = \sigma_{r_2} = \sigma_{r_3} = 300$ feet, $\sigma_{s_1} = \sigma_{s_2} = 100$ feet, $\sigma_{s_3} = 0$ feet.

FIG. 17 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA MUTUALLY VISIBLE TO SATELLITES STATIONED ABOVE 5° EAST LONGITUDE AND 65° WEST LONGITUDE - 1



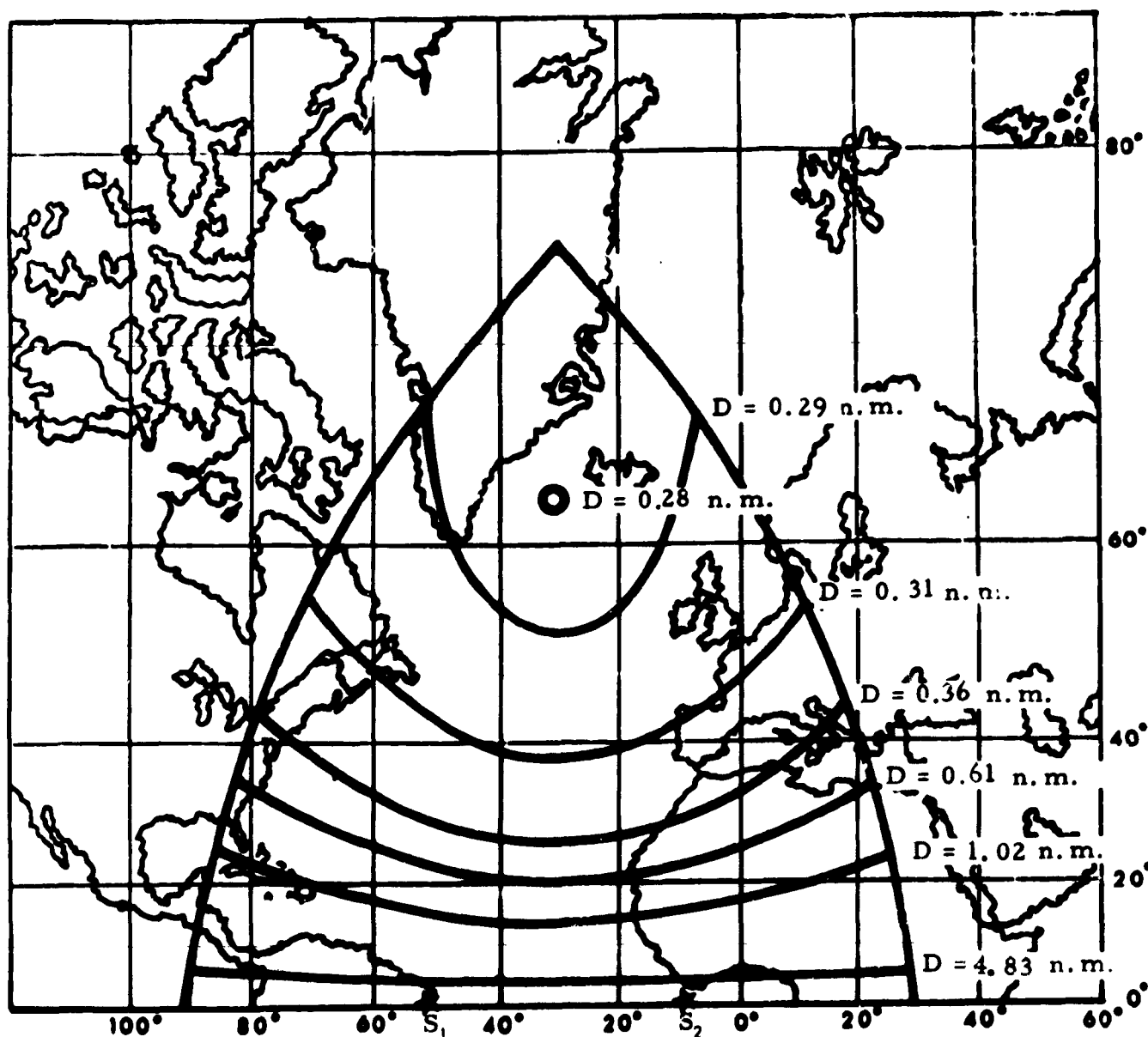
The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming input standard deviations as follows (see text for definitions): $\sigma_{R_1} = \sigma_{R_2} = \sigma_{R_3} = 300$ feet, $\sigma_{S_1} = \sigma_{S_2} = 100$ feet, $\sigma_{S_3} = 0$ feet.

FIG. 18 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA MUTUALLY VISIBLE TO SATELLITES STATIONED ABOVE 0° WEST LONGITUDE AND 60° WEST LONGITUDE - I



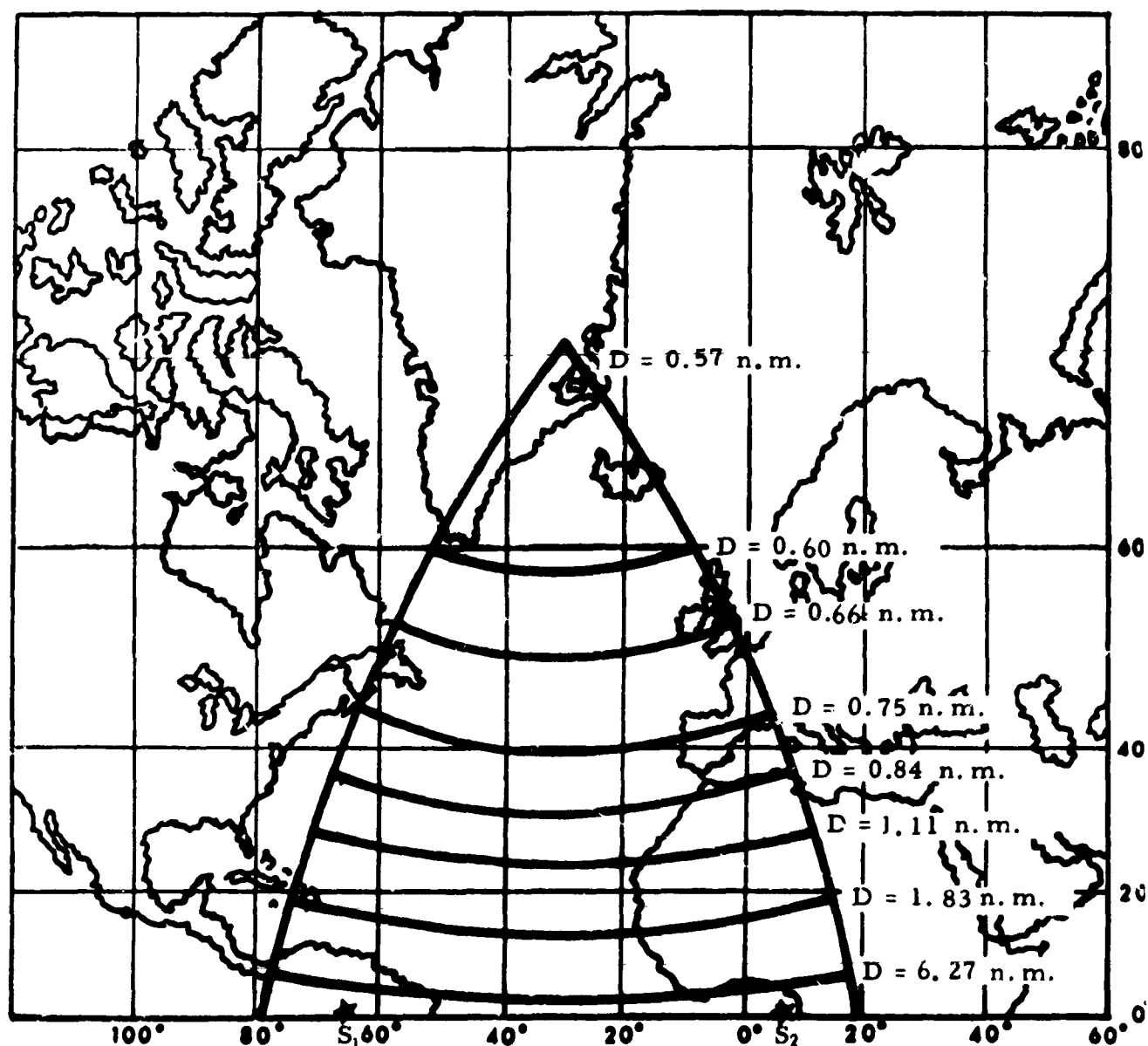
The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming input standard deviations as follows (see text for definitions): $\sigma_{R_1} = \sigma_{R_2} = \sigma_{R_3} = 300$ feet, $\sigma_{S_1} = \sigma_{S_2} = 100$ feet, $c_{S_3} = 0$ feet.

FIG. 19 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA MUTUALLY VISIBLE TO SATELLITES STATIONED ABOVE 50° WEST LONGITUDE AND 55° WEST LONGITUDE - I



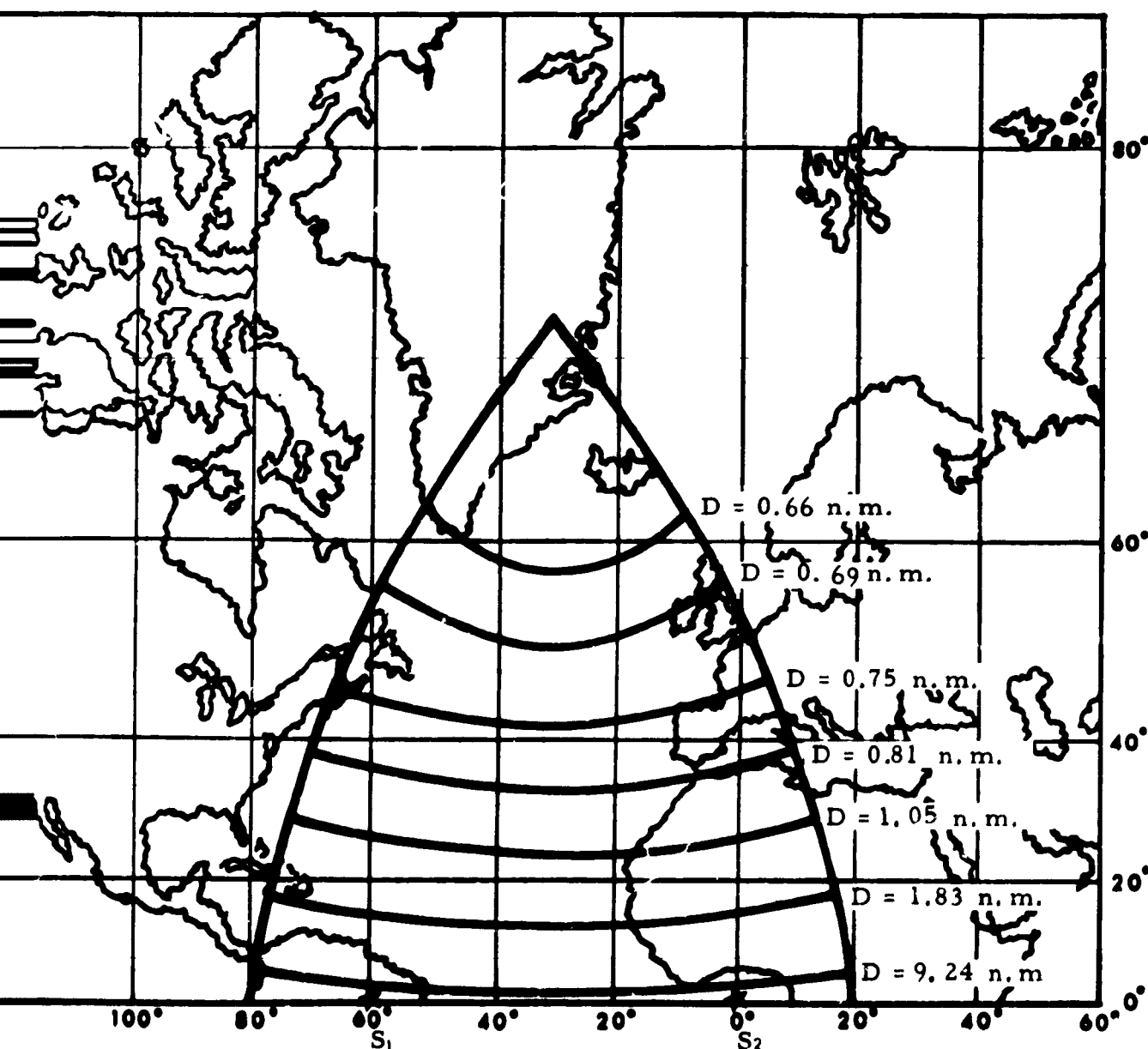
The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming input standard deviations as follows (see text for definitions): $\sigma_{R_1} = \sigma_{R_2} = \sigma_{R_3} = 300$ feet, $\sigma_{S_1} = \sigma_{S_2} = 100$ feet, $\sigma_{S_3} = 0$ feet.

FIG. 20 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA MUTUALLY VISIBLE TO SATELLITES STATIONED ABOVE 10° WEST LONGITUDE AND 50° WEST LONGITUDE - I



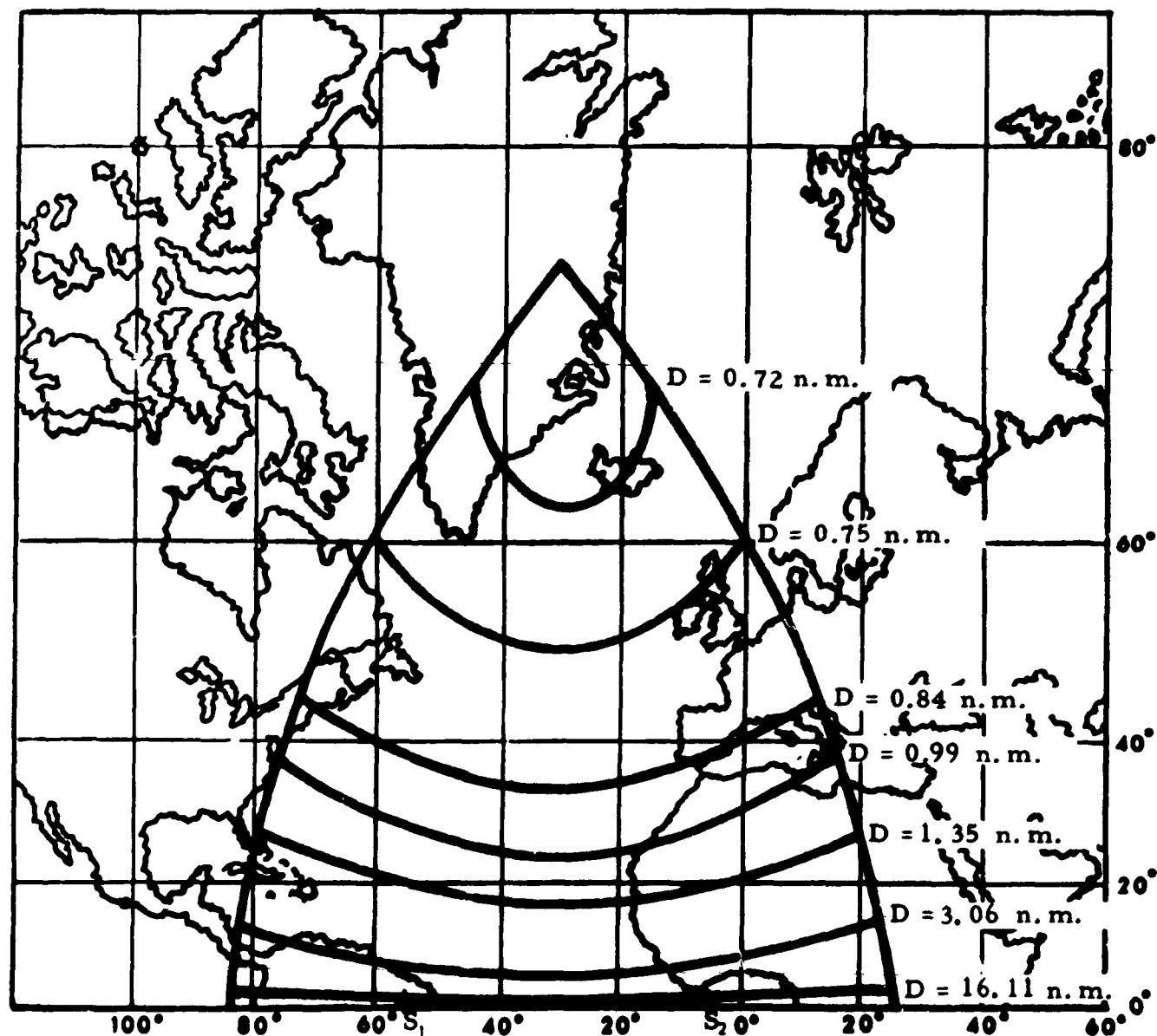
The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming input standard deviations as follows (see text for definitions): $\sigma_{r_1} = \sigma_{r_2} = \sigma_{r_3} = 900$ feet, $\sigma_{s_1} = \sigma_{s_2} = 300$ feet, $\sigma_{s_3} = 0$ feet.

FIG. 21 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA MUTUALLY VISIBLE TO SATELLITES STATIONED ABOVE 5° EAST LONGITUDE AND 65° WEST LONGITUDE - II



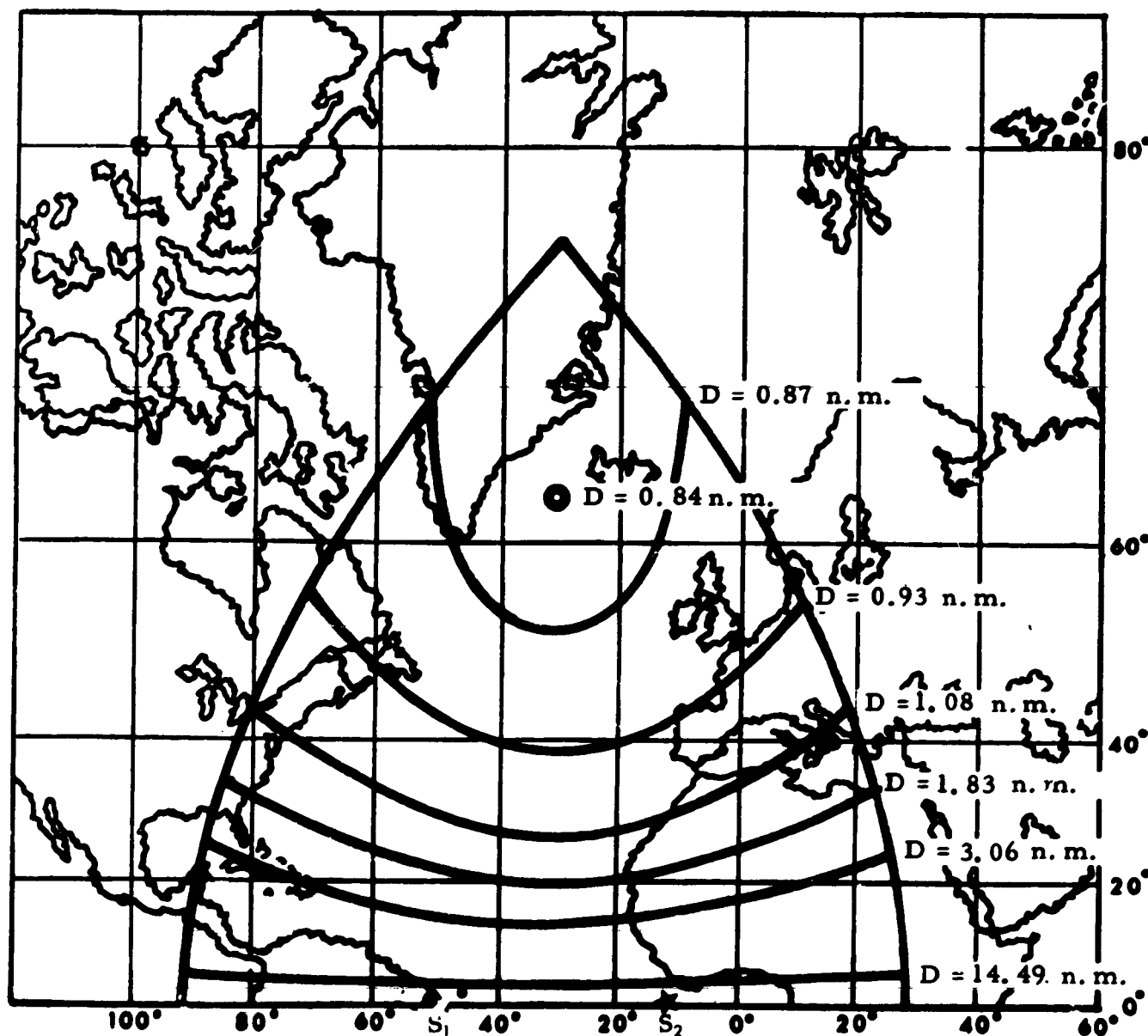
The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming input standard deviations as follows (see text for definitions): $\sigma_{R_1} = \sigma_{R_2} = \sigma_{R_3} = 900$ feet, $\sigma_{S_1} = \sigma_{S_2} = 300$ feet, $\sigma_{S_3} = 0$ feet.

FIG. 22 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA MUTUALLY VISIBLE TO SATELLITES STATIONED ABOVE 0° WEST LONGITUDE AND 60° WEST LONGITUDE - II



The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming input standard deviations as follows (see text for definitions): $\sigma_{R_1} = \sigma_{R_2} = \sigma_{R_3} = 900$ feet, $\sigma_{S_1} = \sigma_{S_2} = 300$ feet, $\sigma_{S_3} = 0$ feet.

FIG. 23 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA MUTUALLY VISIBLE TO SATELLITES STATIONED ABOVE 5° WEST LONGITUDE AND 55° WEST LONGITUDE - II



The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming input standard deviations as follows (see text for definitions): $\sigma_{r_1} = \sigma_{r_2} = \sigma_{r_3} = 900$ feet, $\sigma_{s_1} = \sigma_{s_2} = 300$ feet, $\sigma_{s_3} = 0$ feet.

FIG. 24 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA MUTUALLY VISIBLE TO SATELLITES STATIONED ABOVE 10° WEST LONGITUDE AND 50° WEST LONGITUDE - II

Since the earth's sphere is much smaller than the spheres generated by either R_1 or R_2 , a useful approximation can be made by assuming that the portion of the sphere $R_1 = \text{constant}$, near S'_1 , is a plane. Fig. 25 shows the approximation.

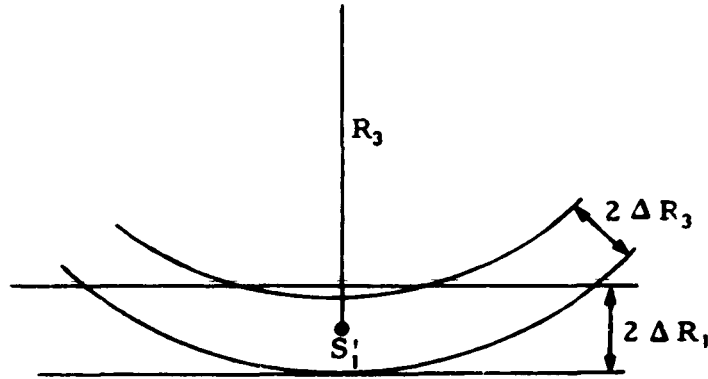


FIG. 25 THE REGION ABOUT THE POINT S'_1 SHOWING MEASUREMENT DEVIATIONS ΔR_1 AND ΔR_3

In Fig. 25 ΔR_1 , ΔR_3 are deviations in the measures of R_1 and R_3 , respectively.

For $\sigma_{R_1} = 300$ feet and $\sigma_{S_1} = 100$ feet

$$\Delta R_1 = 2.2 \sqrt{\sigma_{R_1}^2 + \sigma_{S_1}^2}$$

$$\Delta R_1 = 2.2 \sqrt{(300)^2 + (100)^2}$$

$$\Delta R_1 = 695 \text{ feet}$$

The probability is .972 that the measurement of R_1 is within 695 feet of the true value of R_1 .

For $\sigma_{R_3} = 300$ feet and $\sigma_{S_3} = 0$

$$\Delta R_3 = 2.2 \sqrt{\sigma_{R_3}^2 + \sigma_{S_3}^2}$$

$$\Delta R_3 = 660 \text{ feet}$$

The probability is .972 that the measurement on R_3 is within 660 feet of the true value of R_3 . The probability is .95 that the measurement of position at S'_1 is within ΔR_1 and within ΔR_3 of the true values of R_1 and R_3 . The probability is greater than .95 that the measurement is within the distance d of the true position where d is shown in Fig. 26.

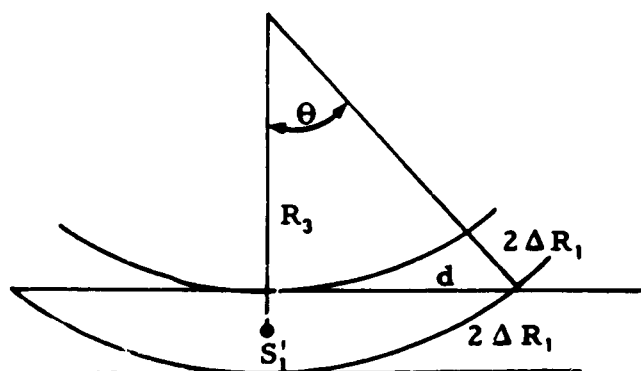


FIG. 26 THE DISTANCE d SHOWS THE LARGEST LINEAR DISTANCE PERPENDICULAR TO R_3 (AND R_1)

From Fig. 26:

$$\cos \theta = \frac{R_3 - \Delta R_1}{R_3 + \Delta R_1}$$

$$\sin \theta = \frac{d}{R_3 + \Delta R_1}$$

$$d = (R_3 + \Delta R_1) (1 - \cos^2 \theta)^{1/2}$$

$$d = (R_3 + \Delta R_1) \left[1 - \left(\frac{R_3 - \Delta R_1}{R_3 + \Delta R_1} \right)^2 \right]^{1/2}$$

$$d = \left[(R_3 + \Delta R_1)^2 - (R_3 - \Delta R_1)^2 \right]^{1/2}$$

$$d = \left[R_3^2 + 2 R_3 \Delta R_1 + \Delta R_1^2 - R_3^2 + 2 R_3 \Delta R_1 - \Delta R_1^2 \right]^{1/2}$$

$$d = 2 \sqrt{R_3 \Delta R_1}$$

$$d = 2 \sqrt{4000 \times .132}$$

$$d = 39.8 \text{ nautical miles}$$

For $\sigma_{R_1} = 300$ feet and $\sigma_{S_1} = 100$ feet the probability is greater than .95 that the measured position is within 39.8 nautical miles of the true position when the true position is the subsatellite point S_1 . It should be emphasized that the value is an upper bound and that real values can be determined only by experiment.

aircraft, a position can be determined (for details see reference [5]).

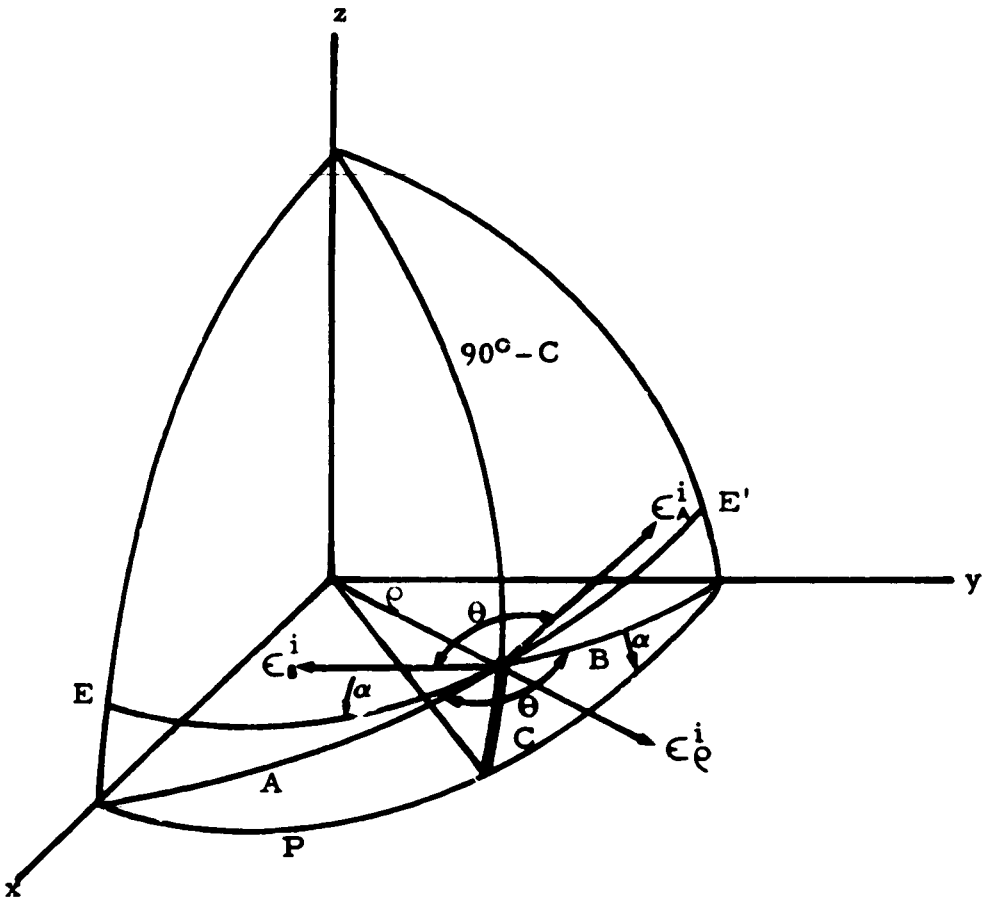


FIG. 27 AN INTERFEROMETER SYSTEM MEASURING THE ANGLES A AND B AND THE DISTANCE ρ

Fig. 27 is a sketch of an orthogonal interferometer system with the interferometer arms along the x and y axes and the z axis orthogonal to both arms. The errors shown are linear errors with $\epsilon_A^i = \rho(\Delta A)^i$ and $\epsilon_B^i = \rho(\Delta B)^i$ where $(\Delta A)^i$ and $(\Delta B)^i$ are angular errors in A and B, respectively. The coordinate set of the error components is not orthogonal making it necessary to apply the T matrix of the probabilistic system model developed in the General Theory. To use the T matrix, it is necessary to identify system angles with angles of the T matrix as follows:

- Identify Φ as the angle between ϵ_B^i and $\epsilon_\theta^i = 90^\circ$
- Identify ψ as the angle between ϵ_θ^i and $\epsilon_A^i = 90^\circ$
- Since Φ and ψ are both 90° , the angle Θ is the angle between ϵ_A^i and ϵ_θ^i . The T matrix,

$$T = \begin{bmatrix} \csc \Phi & 0 & -\cot \Phi \\ -\csc \Phi \cot \Theta & \csc \psi \csc \Theta & \cot \Phi \cot \Theta - \cot \psi \csc \Theta \\ 0 & 0 & 1 \end{bmatrix}$$

is therefore:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ -\cot \Theta & \csc \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

thus:

$$\begin{bmatrix} \epsilon_{x_1}^i \\ \epsilon_{x_2}^i \\ \epsilon_{x_3}^i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\cot \Theta & \csc \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_\theta^i \\ \epsilon_A^i \\ \epsilon_\phi^i \end{bmatrix}$$

and

$$\begin{aligned} \epsilon_{x_1}^i &= \epsilon_\theta^i \\ \epsilon_{x_2}^i &= -\epsilon_\theta^i \cot \Theta + \epsilon_A^i \csc \Theta \\ \epsilon_{x_3}^i &= \epsilon_\phi^i \end{aligned}$$

are the error components in the orthogonal coordinate set (x_1, x_2, x_3) where x_1 is along B, x_3 is along ϕ and x_2 is orthogonal to both x_1 and x_3 as shown in Fig. 28.

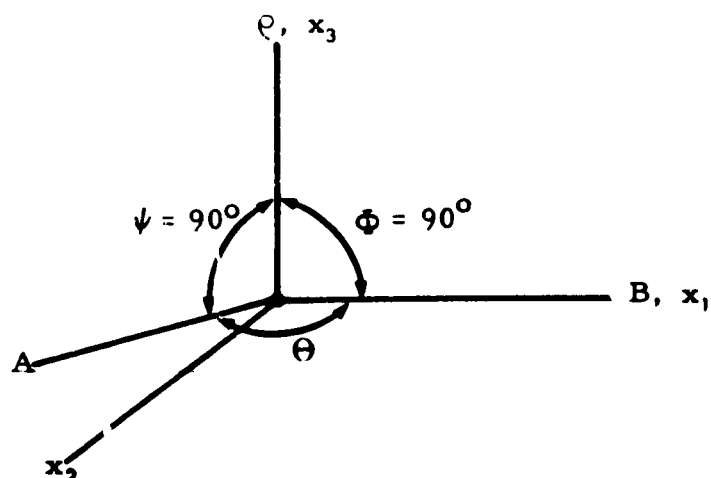


FIG. 28 THE ORTHOGONAL SET (x_1, x_2, x_3) RELATED TO THE NON-ORTHOGONAL SET (Q, A, B)

In Fig. 27 the earth's center is located somewhere along the z axis, and the earth's sphere intersects the spherical segment of Fig. 27 in the arc $\overline{EE'}$. It is evident that none of the orthogonal error components $\epsilon_{x_1}^i$, $\epsilon_{x_2}^i$, $\epsilon_{x_3}^i$ is in the tangent plane of the earth's surface. To get $\epsilon_{x_1}^i = \epsilon_{y_1}^i$ in the tangent plane of the earth's surface, the system must be rotated counterclockwise about the Q axis through the angle α shown in Fig. 27.

Then:

$$\begin{bmatrix} \epsilon_{y_1}^i \\ \epsilon_{y_2}^i \\ \epsilon_{y_3}^i \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{x_1}^i \\ \epsilon_{x_2}^i \\ \epsilon_{x_3}^i \end{bmatrix}$$

$$\epsilon_{y_1}^i = \epsilon_{x_1}^i \cos \alpha + \epsilon_{x_2}^i \sin \alpha$$

$$\epsilon_{y_2}^i = \epsilon_{x_1}^i \sin \alpha + \epsilon_{x_2}^i \cos \alpha$$

$$\epsilon_{y_3}^i = \epsilon_{x_3}^i$$

where $\epsilon_{y_1}^i$ is in the tangent plane of the earth as shown in Fig. 29.

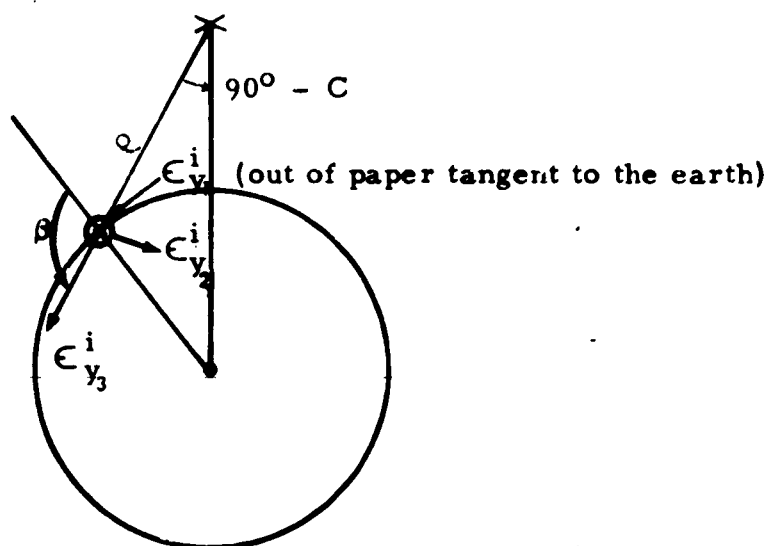


FIG. 29 THE ORTHOGONAL SET ($e_{y_1}^i, e_{y_2}^i, e_{y_3}^i$) WITH $e_{y_1}^i$ IN THE TANGENT PLANE OF THE EARTH

From Fig. 29, $e_{y_2}^i$ can be placed into the tangent plane of the earth by rotating clockwise about the y_1 axis through the angle β . Thus:

$$\begin{bmatrix} e_{z_1}^i \\ e_{z_2}^i \\ e_{z_3}^i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} e_{y_1}^i \\ e_{y_2}^i \\ e_{y_3}^i \end{bmatrix}$$

and

$$\begin{aligned} e_{z_1}^i &= e_{y_1}^i \\ e_{z_2}^i &= e_{y_2}^i \cos \beta - e_{y_3}^i \sin \beta \\ e_{z_3}^i &= e_{y_2}^i \sin \beta + e_{y_3}^i \cos \beta \end{aligned}$$

where

$e_{z_1}^i$ and $e_{z_2}^i$ are in the tangent plane of the earth.

$$e_{z_1}^i = e_{y_1}^i$$

$$\epsilon_{z_2}^i = \epsilon_{y_2}^i \cos \beta - \epsilon_{y_3}^i \sin \beta$$

$$\epsilon_{z_1}^i = \epsilon_{x_1}^i \cos \alpha + \epsilon_{x_2}^i \sin \alpha$$

$$\epsilon_{z_2}^i = (-\epsilon_{x_1}^i \sin \alpha + \epsilon_{x_2}^i \cos \alpha) \cos \beta - \epsilon_{x_3}^i \sin \beta$$

$$\epsilon_{z_1}^i = \epsilon_B^i (\cos \alpha - \cot \theta \sin \alpha) + \epsilon_A^i \csc \theta \sin \alpha$$

$$\begin{aligned} \epsilon_{z_2}^i &= \epsilon_B^i (-\sin \alpha \cos \beta - \cot \theta \cos \alpha \cos \beta) \\ &\quad + \epsilon_A^i \csc \theta \cos \alpha \cos \beta \\ &\quad - \epsilon_C^i \sin \beta \end{aligned}$$

$$\begin{aligned} \sigma_{z_1}^2 &= \frac{1}{N} \sum_{i=1}^N (\epsilon_{z_1}^i)^2 \\ &= \sigma_B^2 (\cos \alpha - \cot \theta \sin \alpha)^2 + \sigma_A^2 (\csc \theta \sin \alpha)^2 \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma_{z_2}^2 &= \sigma_B^2 (-\sin \alpha - \cos \alpha \cot \theta)^2 \cos^2 \beta \\ &\quad + \sigma_A^2 (\csc \theta \cos \alpha \cos \beta)^2 \\ &\quad + \sigma_C^2 \sin^2 \beta \end{aligned}$$

$$d_{rms}^2 = \sigma_{z_1}^2 + \sigma_{z_2}^2$$

$$\begin{aligned} d_{rms}^2 &= \sigma_B^2 \left[(\cos \alpha - \sin \alpha \cot \theta)^2 + (-\sin \alpha - \cos \alpha \cot \theta)^2 \cos^2 \beta \right] \\ &\quad + \sigma_A^2 \left[\csc^2 \theta \sin^2 \alpha + \csc^2 \theta \cos^2 \alpha \cos^2 \beta \right] \\ &\quad + \sigma_C^2 \left[\sin^2 \beta \right] \end{aligned}$$

For a measurement to have a probability greater than .95 of being within some distance D of the true position,

$$D = 2.5 \, d_{rms}$$

Then:

$$D = 2.5 \left\{ \sigma_B^2 \left[(\cos \alpha - \sin \alpha \cot \theta)^2 + (-\sin \alpha - \cos \alpha \cot \theta)^2 \cos^2 \beta \right] + \sigma_A^2 \left[\csc^2 \theta \sin^2 \alpha + \csc^2 \theta \cos^2 \alpha \cos^2 \beta \right] + \sigma_C^2 \left[\sin^2 \beta \right] \right\}^{\frac{1}{2}} \quad (8)$$

To get a mapping of error isograms (constant D curves) on the earth's surface, it is necessary to relate the angles θ , α , β to a system of earth angles. From Fig. 27, using spherical trigonometry:

$$\cos 90^\circ = \cos A \cos B + \sin A \sin B \cos \theta$$

$$\cos \theta = - \frac{\cos A \cos B}{\sin A \sin B}$$

$$\cos A = \cos P \cos C$$

$$\sin A = \sqrt{1 - \cos^2 C \cos^2 P}$$

$$\cos B = \cos (90^\circ - P) \cos C$$

$$\cos B = \sin P \cos C$$

$$\sin B = \sqrt{1 - \sin^2 P \cos^2 C}$$

Therefore:

$$\cos \theta = - \frac{\cos^2 C \sin P \cos P}{\sqrt{1 - \cos^2 C \cos^2 P} \sqrt{1 - \cos^2 C \sin^2 P}} \quad (9)$$

From Fig. 27 the maximum value of C is 90° . The minimum value of C can be derived from Fig. 30.

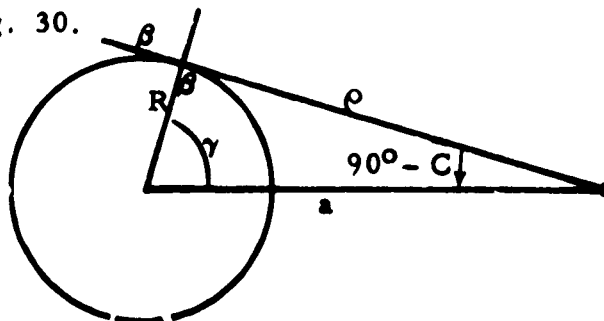


FIG. 30 THE MINIMUM VALUE OF THE ANGLE C IS REACHED WHEN $\beta = 90^\circ$

From Fig. 30 with $\beta = 90^\circ$

$$a^2 = R^2 + \rho^2$$

$$\rho^2 = R^2 + a^2 - 2 a R \cos \gamma$$

$$a^2 - R^2 = R^2 + a^2 - 2 a R \cos \gamma$$

$$\cos \gamma = \frac{R}{a}$$

For a satellite in a synchronous orbit:

$$\cos \gamma \approx \frac{4000}{26300}$$

$$\cos \gamma \approx .151$$

$$\gamma \approx 81^\circ$$

From Fig. 30:

$$90^\circ - C_{min} = 180 - 90 - \gamma$$

$$C_{min} = \gamma$$

$$C_{min} \approx 81^\circ$$

$$\cos C_{min} \approx .151$$

$$\cos^2 C_{min} \approx .023$$

To a very good approximation let $\cos^2 C = 0$ in the denominator of equation (9). That is, in

$$\cos \theta = - \frac{\cos^2 C \sin P \cos P}{\sqrt{1 - \cos^2 C \sin^2 P} \sqrt{1 - \cos^2 C \cos^2 P}}$$

let $\cos^2 C = 0$ in the denominator. Then:

$$\cos \theta = - .023 \sin P \cos P$$

The maximum value of $\cos \theta$ is given for $P = 45^\circ$. Therefore.

$$\cos \theta_{max} = - .023 \left(\frac{1}{2} \right)$$

$$\cos \Theta_{\max} = - .0115$$

$$\Theta_{\max} = 90^{\circ} 40'$$

Therefore:

$$90^{\circ} \geq \Theta \geq 90^{\circ} 40'$$

That is, the value of Θ is very close to 90° over the entire surface of the earth when the satellite is in a synchronous orbit.

In equation (8) for D, the multipliers of $\cot \Theta$ and $\csc \Theta$ are $\sin \alpha$ and $\cos \alpha$. Since the maximum values of these functions is unity to a very good approximation, let

$$\cot \Theta = 0$$

$$\csc \Theta = 1$$

Then:

$$D = 2.5 \left\{ \sigma_s^2 \left[\cos^2 \alpha + \sin^2 \alpha \cos^2 \beta \right] + \sigma_A^2 \left[\sin^2 \alpha + \cos^2 \alpha \cos^2 \beta \right] + \sigma_P^2 \left[\sin^2 \beta \right] \right\}^{\frac{1}{2}}$$

Because the angles A and B are both large over the entire earth surface, assume that $\sigma_A = \sigma_s = \sigma$ then,

$$D = 2.5 \left\{ \sigma^2 \left[1 + \cos^2 \beta \right] + \sigma_P^2 \sin^2 \beta \right\}^{\frac{1}{2}} \quad (10)$$

To a very good approximation the D values are independent of the angle P and the loci of constant D are concentric circles on the earth's sphere centered at the subsatellite point on the earth's sphere.

Reference [5] contains an extensive analysis of factors which affect σ and σ_P in order to arrive at some reasonable estimate for those values. The values assumed in reference [5] are approximately:

$$\sigma = \varrho \times .03 \text{ milliradians}$$

$$\sigma_P = 300 \text{ feet}$$

Since the minimum value of ϱ is 22,300 miles,

$$\sigma = 27,300 \times 5,280 \times .03 \times 10^{-3} \text{ feet}$$

$$\sigma \approx 3,532 \text{ feet (a minimum value)}$$

The maximum influence of σ_ϱ on D in equation (10) is exerted at $\varrho = 90^\circ$. At that point:

$$D = 2.5 (\sigma^2 + \sigma_\varrho^2)^{\frac{1}{2}}$$

Since $\sigma^2 \gg \sigma_\varrho^2$ the influence of σ_ϱ on the system is very slight and system precision is basically a function of the angular errors of the interferometer arms. Assuming σ_ϱ is negligible,

$$D = 2.5 \sigma (1 + \cos^2 \beta)^{\frac{1}{2}} \quad (11)$$

where

$$\sigma = .03 \times 10^{-3} \varrho$$

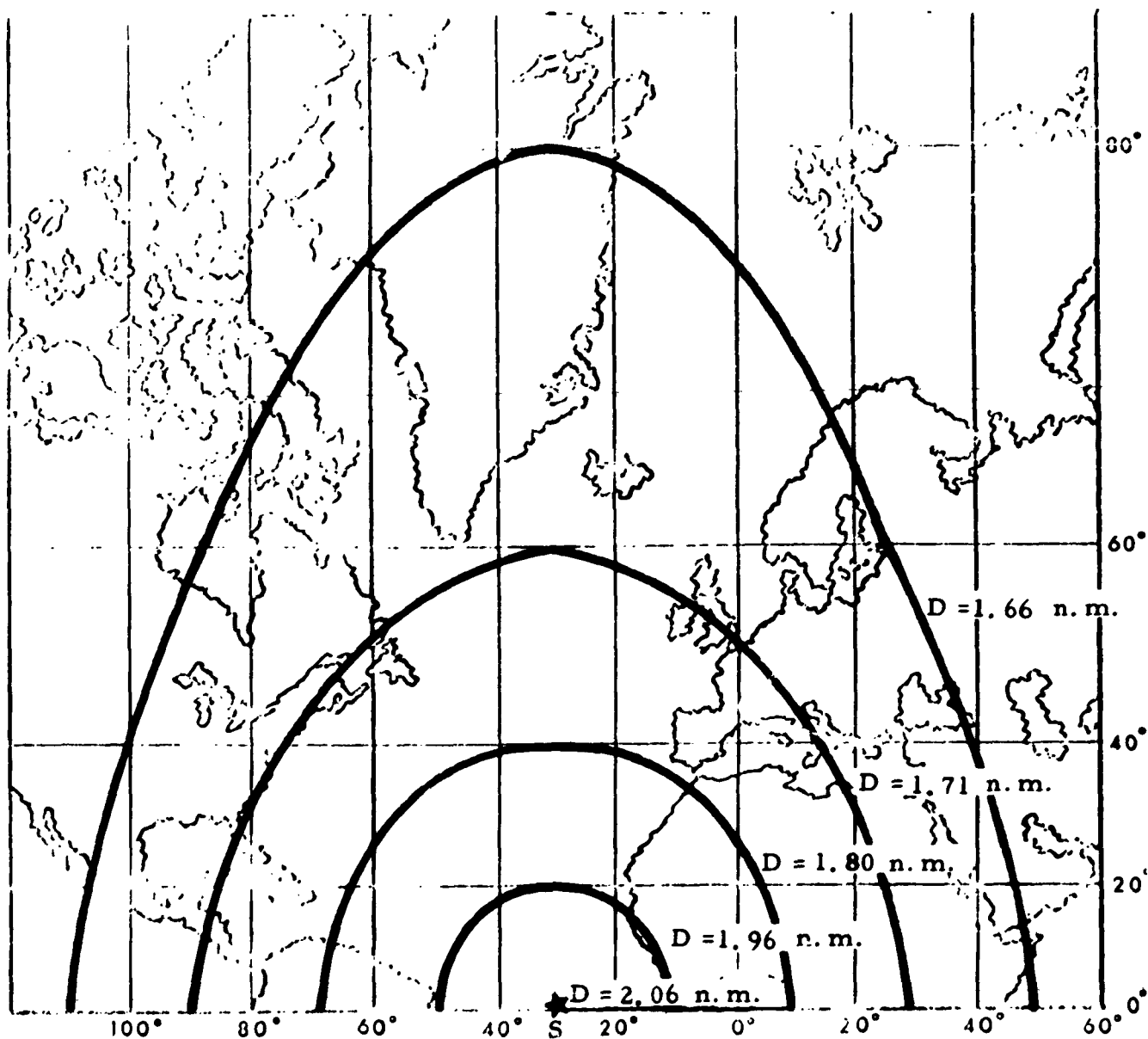
$$\cos \beta = \frac{R - a \cos \gamma}{\varrho}$$

$$\varrho = (a^2 + R^2 - 2 a R \cos \gamma)^{\frac{1}{2}}$$

R, a, and γ are shown in Fig. 30. Constant D values are plotted in Fig. 31. The probability is greater than .95 that the measured position is within D nautical miles (to a very good approximation) of the true position where the satellite is in a synchronous orbit at 30° West Longitude.

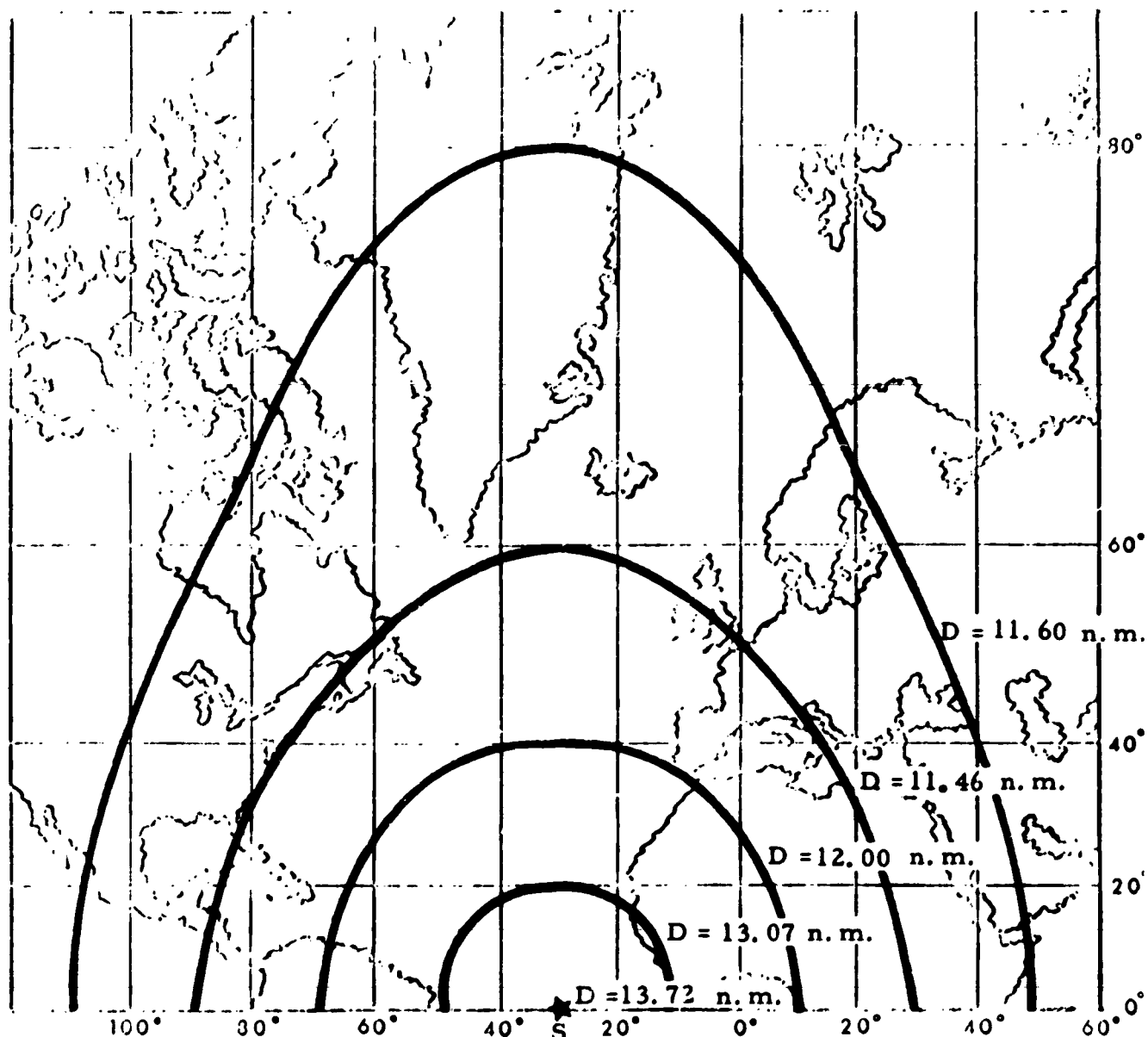
There exists no known experimental data on the performance of an interferometer in space, such as the system proposed here, or even of a spaceborne platform which measures a single angle. However, some experimental data exists on the accuracy of a ground-based interferometer system in use at NAFEC (National Aviation Facilities Experimental Center) of the Federal Aviation Agency located at Atlantic City, New Jersey. MOPTAR (Multi-Object Phase Tracking and Ranging) built by the Cubic Corporation uses interferometer arms which are 125 feet long. The angular accuracy claimed and corroborated by testing at NAFEC for absolute position determination is 200 parts per million in direction cosine (reference [12]).* For angles near 90° , 200 parts per million in direction cosine is equivalent to 0.2 milliradians. The estimate of .03 milliradians for a 200-foot long spaceborne interferometer seems very optimistic. Assuming that the tolerances could be held on a spaceborne 125-foot interferometer system and the angular deviation for the system is 0.2 milliradians, the equation (11) gives constant D values as shown in Fig. 32.

* An integral part of the measuring technique consisted of calibration of the system against fixed, surveyed targets before and after each test run.



The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming an input angular standard deviation of .03 milliradians.

FIG. 31 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA VISIBLE TO A SATELLITE STATIONED ABOVE 30° WEST LONGITUDE



The probability is greater than .95 that a measured position is within D nautical miles of the true position assuming an input angular standard deviation of .2 milliradians (experimental value of the NAFEC MOPTAR installation).

FIG. 32 ERROR ISOGRAMS (CONSTANT D VALUES) FOR THE NORTH ATLANTIC AREA VISIBLE TO A SATELLITE STATIONED ABOVE 30° WEST LONGITUDE - 125-FOOT INTERFEROMETER

DISCUSSION

The position determination probabilistic system model developed in this report is a useful computational tool with the proper identification of the model angles (Φ , ψ , θ angles of the T matrix) to the angles of any particular position determination system. Once the identification is made, the use of the model leads to a solution in a straightforward manner and results are presented in the form of the physical size and orientation of the trivariate normal equi-probability error density ellipsoid. The model was but partially used in the Applications section of this report, because the nature of the solution sought allowed the use of the drms error statistic which eased the computational problem. It should be emphasized, however, that the model does include the mathematical methodology necessary for getting the more exact physical picture of position determination system errors in the form of the trivariate normal equi-probability error density ellipsoid and its orientation in three-dimensional space, when system considerations warrant the computational work involved. The model makes use of a first-order approximation in the system geometry by assuming that for small distances from the true position quadric surfaces of a position determination system are planes. In applying the model to the two-satellite RHO-RHO-RHO system a singularity was found at the equator (i.e., $D = \text{infinity}$). A second-order approximation was required to get an idea of system quality at the singularity. The mapping of constant D values for areas other than near the equator shows that the model gives a good approximation to the system errors. Because the one-satellite RHO-THETA-THETA system quadric surfaces are cones with very large apex angles (162° to 180°), the divergence of a plane surface from the true conical surface is extremely slight. A simple calculation shows that for D values less than 25 miles, the plane does not diverge from the cone by more than 25 feet. The application of the model to the one-satellite system gives an excellent approximation to system errors with no singularities.

A report of the Requirements Panel of the Joint Navigation Satellite Committee dated February 1965, recommends aircraft position determination for a controller to an accuracy of 5 nautical miles in 1965, improving to 3 nautical miles in 1975 and to 1 nautical mile by 2000.

The predicted D values for the RHO-RHO-RHO technique indicate that operationally practical values of positional accuracy can be achieved for attainable instrumental errors in the system. In the heavy traffic density area between Newfoundland and Ireland, excellent results can be expected even for the more conservative estimates of input standard deviations. Near the equator, where the confidence level of the system is low, the traffic density is also low and operational solutions to the air

traffic problem will suffice for a longer time than the operational lifetime of a first-generation satellite system. The practical aspects of the system are such that little research and development is required for any needed technical innovations or problems. Indeed, the most difficult problems for system implementation for the two-satellite system would be economic and administrative rather than technical. As with any new system, some learning and experience would be necessary to solve operational problems, but the obstacles seem to be very small and should not limit the efficient use of the system. Because of the excellent results which can be expected in the heavy traffic density area, and the system simplicity, as well as the simplicity of the satellites, a two-satellite system would be a good choice for the first-generation communication-surveillance satellite system for the North Atlantic Air Traffic Control Area. The choice of satellite separation would depend upon functional interface between the communication subsystem and the surveillance subsystem and upon the emphasis placed upon the possibility of system expansion for worldwide coverage, since the variation in system precision is not very dependent upon the satellite separation. Future studies of the Systems Research and Development Service on satellite systems should be largely concerned with the solution of the problem of satellite separation as well as other facets of the communication-surveillance satellite system design.

If the input angular standard deviation of .03 milliradians can be achieved for the one-satellite RHO-THETA-THETA technique, the analysis indicates that operationally practical values of positional accuracy should be achieved. Available experimental evidence on the performance of a ground-based interferometer (the MOPTAR installation at NAFEC) indicates that the estimate is optimistic. A 125-foot interferometer satellite system with the precision of the ground-based interferometer at NAFEC gives the somewhat marginal D values of Fig. 32. Whether or not a significant increase in system precision can be attained by longer interferometer arms in the hostile space environment is debatable. Estimates could be in error by an order of magnitude or more. The satellite system is simple, consisting of a single satellite between 30° and 40° West Longitude but the satellite itself requires considerable research and development. The two most serious problems are satellite attitude stabilization and automatic boom construction with antenna sensors strategically distributed along the interferometer arm to obviate ambiguities. The problems are not isolated from one another. Attempting to increase boom length to increase system precision exaggerates the attitude stabilization problem. The National Aeronautics and Space Administration is planning three experiments on gravity gradient stabilization with two of them at synchronous orbit altitude. The ATS (Advanced Technological Satellites) series is planned to run from late 1966 to late 1968. The research and development required on the techniques used by an interferometer-type satellite system, indicates that from the point of view of operational traffic control satellites, the system should be considered a second-generation system if a reasonable system precision can be achieved in the future.

CONCLUSIONS

It is concluded that:

1. A determination of theoretical system quality of position determination systems can be made by the application of the mathematical model constructed in this report.
2. Error isograms for the two-satellite surveillance systems predict excellent results for determining aircraft position in the heavy traffic area over the North Atlantic and less precise measurements in a narrow band near the equator.
3. Experimental evidence on the NAFEC MOPTAR (an interferometer with orthogonal arms) does not support the most optimistic error estimates for the interferometer technique. The research and development still required on the techniques used by an interferometer-type satellite system, indicate that attainment of the requisite precision by use of the one-satellite technique will require more time and effort than by use of the two satellite multi-rho technique.

RECOMMENDATIONS

It is recommended that:

1. The probabilistic position determination system model constructed in this report be further exploited by the Systems Research and Development Service for system studies as required in the area of air traffic surveillance and navigation.
2. The Systems Research and Development Service explore in more detail the two-satellite multi-rho surveillance technique to (a) determine the feasibility of adapting the technique for use with a VHF communication satellite and (b) formulate a system, utilizing the technique, which can be subjected to rigorous technical, operational and economic analyses to determine its utility in the North Atlantic Air Traffic Control operation.
3. The Systems Research and Development Service continue in-house study of the interferometer-type satellite technique and also provide inputs to experimentation by NASA and others. Consideration of the interferometer technique and comparison with other techniques should be continued, since the technique, if proven, could offer certain advantages in a later-generation satellite surveillance system.

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